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Linear and Parabolic Motion - Lesson 6

Projectile Motion

LI

- Derive and use the Equations of Projectile Motion.

SC

- Vector calculus.

Assumptions

- No air resistance.
- Constant gravity.
- Motion in the vertical $x - y$ plane only.

$$\underline{\mathbf{r}}(t) = x(t)\,\underline{\mathbf{i}} + y(t)\,\underline{\mathbf{j}}$$

For brevity, this is sometimes written,

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}}$$

Equations of Motion

Fired from Origin

$$\underline{F} = m\,\underline{g}$$

$$m\,\ddot{\underline{r}} = m\,\underline{g}$$

$$\ddot{\underline{r}} = -g\,\underline{j} \qquad \left(\begin{array}{l} \underline{g} = -g\,\underline{j} \\ g = 9.8\ \mathrm{m\,s^{-2}} \end{array}\right)$$

$$\ddot{x}\,\underline{i} + \ddot{y}\,\underline{j} = -g\,\underline{j}$$

$$\ddot{x} = 0 \qquad\qquad \ddot{y} = -g$$

$$\dot{x} = C \qquad\qquad \dot{y} = -g\,t + E$$

When $t = 0$, $\dot{x} = v\cos\theta$;
so, $C = 0$.

$$\dot{x} = v\cos\theta$$

$$x = (v\cos\theta)\,t + D$$

When $t = 0$, $x = 0$;
so, $D = 0$.

$$\boxed{x(t) = (v\cos\theta)\,t}$$

When $t = 0$, $\dot{y} = v\sin\theta$;
so, $E = v\sin\theta$.

$$\dot{y} = -g\,t + v\sin\theta$$

$$y = -(1/2)\,g\,t^2 + (v\sin\theta)\,t + F$$

When $t = 0$, $y = 0$;
so, $F = 0$.

$$\boxed{y(t) = (v\sin\theta)\,t - (1/2)\,g\,t^2}$$

These equations can be written as,

$$\boxed{\begin{array}{c} x(t) = v_x\,t \\ (v_x = v\cos\theta) \end{array}} \qquad \boxed{\begin{array}{c} y(t) = v_y\,t - (1/2)\,g\,t^2 \\ (v_y = v\sin\theta) \end{array}}$$

Fired from Elsewhere

If the projectile is fired from (x_o, y_o), the equations become,

$$\boxed{x(t) = (v\cos\theta)\,t + x_o} \qquad \boxed{y(t) = (v\sin\theta)\,t - (1/2)\,g\,t^2 + y_o}$$

Trajectory Equation

The Trajectory Equation describes y as a function of x (the variable t is eliminated).

$$x(t) = (v\cos\theta)\,t \qquad (1)$$

$$y(t) = (v\sin\theta)\,t - (1/2)\,g\,t^2 \qquad (2)$$

$$(1) \Rightarrow \quad t = \frac{x}{v\cos\theta}$$

$$(2) \Rightarrow \quad y = v\sin\theta\left(\frac{x}{v\cos\theta}\right) - \frac{g}{2}\left(\frac{x}{v\cos\theta}\right)^2$$

$$\Rightarrow \quad y = x\tan\theta - \frac{g\,x^2}{2\,v^2}\left(\frac{1}{\cos^2\theta}\right)$$

$$\Rightarrow \quad y = x\tan\theta - \frac{g\,x^2}{2\,v^2}\sec^2\theta$$

$$\Rightarrow \quad \boxed{y = x\tan\theta - \frac{g\,x^2}{2\,v^2}(1 + \tan^2\theta)}$$

(Trajectory Equation for a particle fired from (0, 0))

$$\boxed{y = (x - x_0)\tan\theta - \frac{g}{2\,v^2}(x - x_0)^2(1 + \tan^2\theta) + y_0}$$

(Trajectory Equation for a particle fired from (x_0, y_0))

Time of Flight and Range

Time of Flight (t_F)

Time taken to get back to $y = 0$.

$$y = (v \sin\theta)\, t - (1/2)\, g\, t^2 \quad , \quad y = 0$$

$$(v \sin\theta)\, t - (1/2)\, g\, t^2 = 0$$

$$t\,(v \sin\theta - (1/2)\, g\, t) = 0$$

$$t = 0, \quad v \sin\theta - (1/2)\, g\, t = 0$$

$$\downarrow$$

$$\boxed{t_F = \frac{2 v \sin\theta}{g}}$$

(Time of Flight for a particle fired from (0, 0))

Range (R)

Distance travelled during t_F.

$$R = (v \cos\theta)\, t_F$$

$$R = (v \cos\theta) \left(\frac{2 v \sin\theta}{g} \right)$$

$$\boxed{R = \frac{2 v^2 \sin\theta \cos\theta}{g} = \frac{v^2 \sin 2\theta}{g}}$$

(Range for a particle fired from (0, 0))

Maximum Range and Angle Required

Maximum range (thinking of range as a function of θ).

$$R(\theta) = \frac{v^2 \sin 2\theta}{g}$$

R is a maximum when $\sin 2\theta$ takes on its maximum value. Maximum of $\sin 2\theta$ is 1. Hence,

$$R_{MAX.} = \frac{v^2 (1)}{g}$$

$$\boxed{R_{MAX.} = \frac{v^2}{g}}$$

(Maximum Range for a particle fired from (0, 0))

R is a maximum when $\sin 2\theta$ is a maximum; this occurs when 2θ is $\pi/2$. Hence,

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

(Angle for $R_{MAX.}$ for a particle fired from (0, 0))

or use calculus

Maximum Height

Time to Max. Height ($t_{MAX.\,H}$)

Time to maximum height is half the time of flight.

$$t_{MAX.\,H} = (1/2)\, t_F$$

$$t_{MAX.\,H} = (1/2)\left(\frac{2\, v \sin\theta}{g}\right)$$

$$\boxed{t_{MAX.\,H} = \frac{v \sin\theta}{g}}$$

(Time to Maximum Height for a particle fired from (0, 0))

or use calculus

Max. Height ($y_{MAX.}$)

$$y_{MAX.} = (v \sin\theta)\, t_{MAX.\,H} - (1/2)\, g\, t_{MAX.\,H}^{\,2}$$

$$y_{MAX.} = (v \sin\theta)\left(\frac{v \sin\theta}{g}\right) - (1/2)\, g\left(\frac{v \sin\theta}{g}\right)^2$$

$$y_{MAX.} = \left(\frac{v^2 \sin^2\theta}{g}\right) - \left(\frac{v^2 \sin^2\theta}{2\, g}\right)$$

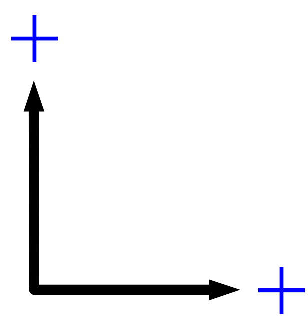
$$\boxed{y_{MAX.} = \frac{v^2 \sin^2\theta}{2\, g}}$$

(Maximum Height for a particle fired from (0, 0))

The equations on the previous 3 pages do not need to be memorised (although it's useful to do so !), but their derivations should be known.

In actual questions, it is sometimes best to use mechanics principles to work out whatever is needed by analysing horizontal and vertical motion separately.

We will agree to use the following sign convention for all projectile motion questions in this lesson :



Example 1

A particle is projected from a point on a horizontal plane and has an initial velocity of $28\sqrt{3}$ m/s at an angle of elevation of 60°.

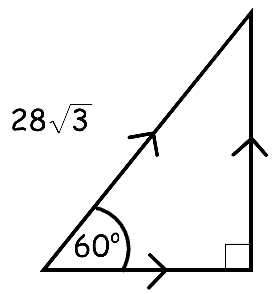
Show that the time taken to reach the greatest height is 30/7 seconds and find the greatest height reached.

First, write the equations for horizontal and vertical displacement :

$$x(t) = v_x t$$

$$y(t) = v_y t - (1/2) g t^2$$

Next use the velocity triangle to obtain the horizontal and vertical components of velocity at the point of projection :



$$v_y(0) = 28\sqrt{3} \sin 60^\circ = 42$$

$$v_x(0) = 28\sqrt{3} \cos 60^\circ = 14\sqrt{3}$$

Hence,

$$x(t) = 14\sqrt{3}\, t \quad \Rightarrow \quad v_x(t) = 14\sqrt{3}$$

$$y(t) = 42t - (1/2) g t^2 \quad \Rightarrow \quad v_y(t) = 42 - 9.8t$$

The greatest height $y_{MAX.}$ is reached when $v_y(t) = 0$. So,

$$42 - 9.8t = 0$$

$$\Rightarrow \quad t = 42/9.8$$

$$\Rightarrow \quad \boxed{t = 30/7 \text{ s}}$$

$$y_{MAX.} = y(30/7)$$

$$\Rightarrow \quad y_{MAX.} = 42(30/7) - (4.9)(30/7)^2$$

$$\Rightarrow \quad \boxed{y_{MAX.} = 90 \text{ m}}$$

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Example 2

A particle is projected from an origin O with a velocity of $(30\,\underline{i} + 40\,\underline{j})$ m/s.

Find the position and velocity vectors of the particle 5 seconds after projection.

Hence find the distance of the particle from O and the speed and direction of its motion at this time.

$$x(t) = 30t \quad \Rightarrow \quad v_x(t) = 30$$

$$y(t) = 40t - (1/2)\,g\,t^2 \quad \Rightarrow \quad v_y(t) = 40 - 9.8t$$

$$x(5) = 30(5) = 150$$

$$y(5) = 40(5) - (1/2)(9.8)(5)^2 = 77.5$$

$$\therefore \quad \boxed{\underline{r}(5) = 150\,\underline{i} + 77.5\,\underline{j}}$$

$$v_x(5) = 30$$

$$v_y(5) = 40 - 9.8(5) = -9$$

$$\therefore \quad \boxed{\underline{v}(5) = 30\,\underline{i} - 9\,\underline{j}}$$

$$r(5) = \sqrt{150^2 + 77.5^2}$$

$$\Rightarrow \quad r(5) = 168.83\ldots$$

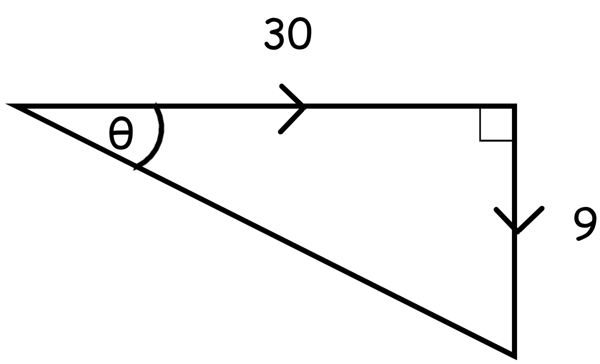
$$\Rightarrow \quad \boxed{r(5) = 168.8 \text{ m} \ (1 \text{ d.p.})}$$

$$v(5) = \sqrt{30^2 + (-9)^2}$$

$$\Rightarrow \quad v(5) = 31.32\ldots$$

$$\Rightarrow \quad \boxed{v(5) = 31.3 \text{ m/s} \ (1 \text{ d.p.})}$$

Direction of motion is the angle the velocity vector makes with either the horizontal or vertical (horizontal is clearly better here) :



$$\theta = \tan^{-1}(9/30)$$

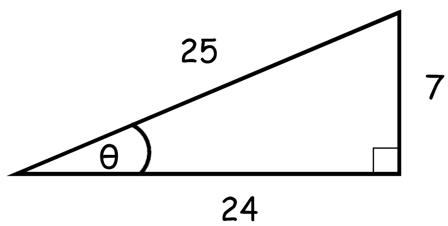
$$\theta = 16.7^\circ \ (1 \text{ d.p.})$$

$$\boxed{\text{Direction of motion is } 16.7^\circ \text{ below the horizontal}}$$

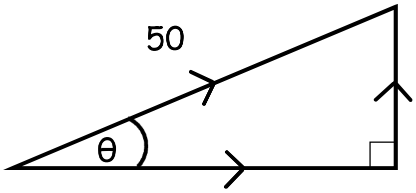
Example 3

A rock is thrown from the edge of a vertical cliff with a velocity of 50 m/s at an angle of $\tan^{-1}(7/24)$ to the horizontal. The stone strikes the sea at a point 240 m from the front of the cliff.

Find the time for which the stone is in the air and the height of the cliff.

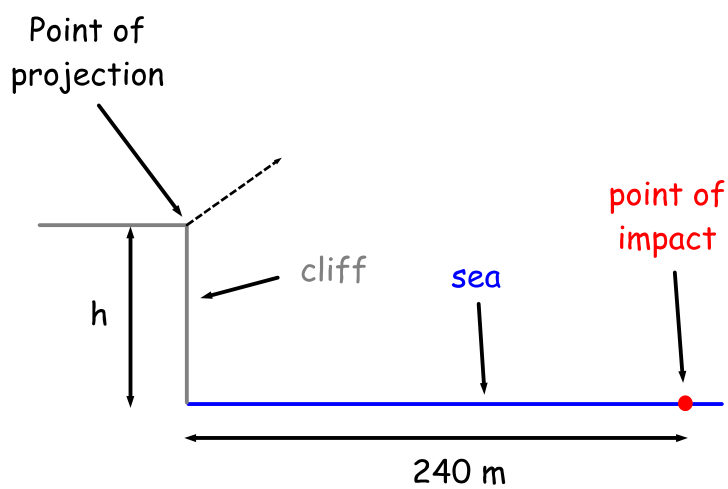


$$\theta = \tan^{-1}(7/24)$$
$$\sin\theta = 7/25$$
$$\cos\theta = 24/25$$



$$v_y(0) = 50\sin\theta = 14$$

$$v_x(0) = 50\cos\theta = 48$$



Taking the origin to be at the point of projection, we have :

$$x(t) = 48\,t \quad \Rightarrow \quad v_x(t) = 48$$

$$y(t) = 14\,t - (1/2)\,g\,t^2 \quad \Rightarrow \quad v_y(t) = 14 - 9.8\,t$$

The strategy is to use the $y(t)$ equation to work out the height, once we know the time taken (i.e. the time of flight), which is obtained from the $x(t)$ equation.

$$48\, t_F = 240$$

$$\Rightarrow \quad \boxed{t_F = 5\ s}$$

As the origin is from the point of projection, the displacement y covered in a time of $5\ s$ is $-h$. So,

$$y(5) = -h$$

$$\Rightarrow \quad h = -y(5)$$

$$\Rightarrow \quad h = (1/2)(9.8)(5)^2 - 14(5)$$

$$\Rightarrow \quad \boxed{h = 52.5\ m}$$

Example 4

A ball is projected with a velocity of 28 m/s at a point on a horizontal plane.

If the range of the ball on the plane is 64 m, find the two possible angles of projection.

Let θ be a possible angle of projection.

$$R = \frac{v^2 \sin 2\theta}{g}$$

$$\therefore \quad 64 = \frac{28^2 \sin 2\theta}{9.8}$$

$$\Rightarrow \quad 784 \sin 2\theta = 627.2$$

$$\Rightarrow \quad \sin 2\theta = 0.8$$

$$\therefore \quad \underline{RA = \sin^{-1}(0.8) \approx 53.1\ldots^\circ}$$

$$\therefore \quad 2\theta = 53.1\ldots^\circ,\ 180 - 53.1\ldots^\circ$$

$$\Rightarrow \quad 2\theta = 53.1\ldots^\circ,\ 126.8\ldots^\circ$$

$$\Rightarrow \quad \boxed{\theta = 26.6^\circ,\ 63.4^\circ}$$

Example 5

A projectile is fired at a speed of 14 m/s and passes through the point (8, 2).

Show that the 2 possible angles of projection are $\tan^{-1}(1/2)$ and $\tan^{-1}(9/2)$.

$$y = x\tan\theta - \frac{g x^2}{2v^2}(1 + \tan^2\theta) \quad , \quad (\overset{x}{8}, \overset{y}{2})$$

$$2 = 8\tan\theta - \frac{9.8(8)^2}{2(14)^2}(1 + \tan^2\theta)$$

$$2 = 8\tan\theta - \frac{49(64)}{2(5)(196)}(1 + \tan^2\theta)$$

$$\boxed{\begin{aligned} g &= 9.8 \\ g &= \frac{98}{10} \\ g &= \frac{49}{5} \end{aligned}}$$

$$2 = 8\tan\theta - \frac{8}{5}(1 + \tan^2\theta)$$

$$10 = 40\tan\theta - 8(1 + \tan^2\theta)$$

$$10 = 40\tan\theta - 8 - 8\tan^2\theta$$

$$0 = 40\tan\theta - 18 - 8\tan^2\theta$$

$$0 = 20\tan\theta - 9 - 4\tan^2\theta$$

$$0 = 4\tan^2\theta - 20\tan\theta + 9$$

$$0 = (2\tan\theta - 1)(2\tan\theta - 9)$$

$$2\tan\theta = 1, \; 2\tan\theta = 9$$

$$\boxed{\theta = \tan^{-1}(1/2), \; \tan^{-1}(9/2)}$$

Example 6

The greatest height reached by a projectile is one-seventh of its range.

Calculate the angle of projection.

$$y_{MAX.} = (1/7)\,R$$

$$R = 7\,y_{MAX.}$$

$$\frac{2\,v^2 \sin\theta\cos\theta}{g} = \frac{7\,v^2 \sin^2\theta}{2\,g}$$

$$4\sin\theta\cos\theta = 7\sin^2\theta$$

$$7\sin^2\theta - 4\sin\theta\cos\theta = 0$$

$$\sin\theta\,(7\sin\theta - 4\cos\theta) = 0$$

$$\sin\theta = 0,\ 7\sin\theta - 4\cos\theta = 0$$

As $0 < \theta < \pi/2$, $\sin\theta \neq 0$. So,

$$7\sin\theta - 4\cos\theta = 0$$

$$7\sin\theta = 4\cos\theta$$

As $0 < \theta < \pi/2$, $\cos\theta \neq 0$. So,

$$\tan\theta = 4/7$$

$$\boxed{\theta = \tan^{-1}(4/7)}$$

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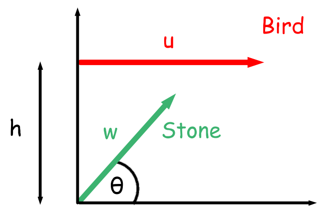
Example 7

A girl throws a stone with speed w m/s at a bird travelling with constant horizontal speed u m/s. When the stone is released, the bird is at a vertical height of h metres.

Show that the stone hits the bird provided that
$w^2 \geq u^2 + 2gh$.



When bird is at height h, stone is thrown at angle θ with speed w (take this moment to be $t = 0$)

Bird	Stone
$x_B = ut$	$x_S = (w\cos\theta)\,t$
$y_B = h$	$y_S = (w\sin\theta)\,t - (1/2)\,g\,t^2$

Stone hits bird provided that the equations $x_B = x_S$ and $y_B = y_S$ are satisfied simultaneously (i.e. for the same t - value).

$$x_B = x_S$$

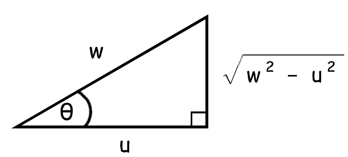
$$ut = (w\cos\theta)\,t$$

Clearly $t \neq 0$. So,

$$u = (w\cos\theta)$$

$$\cos\theta = u/w$$

$$\sin\theta = \frac{\sqrt{w^2 - u^2}}{w}$$



$$y_B = y_S$$

$$h = (w\sin\theta)\,t - (1/2)\,g\,t^2$$

$$h = \left(\sqrt{w^2 - u^2}\right)t - (1/2)\,g\,t^2$$

$$g\,t^2 - 2\left(\sqrt{w^2 - u^2}\right)t + 2h = 0$$

Stone hits bird provided that this equation has a solution; hence, the discriminant of this quadratic must be greater than or equal to 0 :

$$4\,(w^2 - u^2) - 4\,g\,(2h) \geq 0$$

$$w^2 - u^2 - 2gh \geq 0$$

$$\boxed{w^2 \geq u^2 + 2gh}$$

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Blue Book

- pg. 281-282 Ex. 12 A Q 2, 4, 9, 12, 13, 16, 19.
- pg. 288-291 Ex. 12 B Q 1, 8, 9, 13, 16, 19, 28, 32-35.
- pg. 294-296 Ex. 12 C Q 7, 10, 13.