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Unit 2 : Sequences and Series - Lesson 6

Power Series

LI

- Know what a Power Series is.
- Find the sum to infinity of a power series.

SC

- Difference of partial sums technique.
- Sum to infinity of a geometric series.

Jan 10-12:11

An infinite series is convergent if the sum to infinity exists; otherwise it is divergent

A power series is an infinite series of the form :

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where x is regarded as a variable and all the $a_r \in \mathbb{R}$ and do not depend on x

The convergence or divergence of a power series is dependent on the value of x

A power series converges if :

$$\lim_{n \to \infty} \left| \frac{a_{n+1} x}{a_n} \right| < 1$$

Oct 9-15:09

Useful Result

$$\boxed{\text{If } |x| < 1, \text{ as } n \longrightarrow \infty, n x^n \longrightarrow 0}$$

Nov 9-14:42

Example 1

Find the sum to infinity of the series given by
$S = 1 + 3x + 5x^2 + 7x^3 + \ldots + (2n - 1)x^{n-1} + \ldots$
and find for which x values it is convergent.

$$S = 1 + 3x + 5x^2 + 7x^3 + \ldots + (2n - 1)x^{n-1} + \ldots$$

The nth partial sum of S is,

$$S_n = 1 + 3x + 5x^2 + 7x^3 + \ldots + (2n - 1)x^{n-1}$$

$$\therefore \quad xS_n = x + 3x^2 + 5x^3 + \ldots + (2n - 1)x^n$$

$$\Rightarrow \quad (1 - x)S_n = 1 + 2x + 2x^2 + \ldots + 2x^{n-1} - (2n - 1)x^n$$

$$\Rightarrow \quad (1 - x)S_n = 2 + 2x + 2x^2 + \ldots + 2x^{n-1} - (2n - 1)x^n - 1$$

$$\Rightarrow \quad (1 - x)S_n = 2(1 + x + x^2 + \ldots + x^{n-1}) - (2n - 1)x^n - 1$$

$$\Rightarrow \quad (1 - x)S_n = \frac{2(1 - x^n)}{1 - x} - (2n - 1)x^n - 1$$

$$\Rightarrow \quad S_n = \frac{2(1 - x^n) - 1(1 - x)}{(1 - x)^2} - \frac{(2n - 1)x^n}{1 - x}$$

If $|x| < 1$, as $n \to \infty$, $x^n \to 0$ and $nx^n \to 0$; hence,
taking the limit $n \to \infty$ in the above equation gives,

$$S_\infty = \frac{2(1 - 0) - 1(1 - x)}{(1 - x)^2} - 0$$

$$\Rightarrow \quad S_\infty = \frac{2 - 1 + x}{(1 - x)^2}$$

$$\Rightarrow \quad \boxed{S_\infty = \frac{1 + x}{(1 - x)^2} \quad (|x| < 1)}$$

Oct 2-23:10

Example 2

Find the sum to infinity of the series given by $S = 1 + 3x + 7x^2 + 15x^3 + \ldots$ and find for which x values it is convergent.

$$S = 1 + 3x + 7x^2 + 15x^3 + \ldots$$

$$\therefore \quad xS = x + 3x^2 + 7x^3 + 15x^4 + \ldots$$

$$\Rightarrow \quad (1 - x)S = 1 + 2x + 4x^2 + 8x^3 + \ldots$$

The RHS of the above is a geometric series with first term 1 and common ratio $2x$; hence, it will converge when $|2x| < 1$ $\Rightarrow |x| < 1/2$. So,

$$(1 - x)S = \frac{1}{1 - 2x}$$

$$\Rightarrow \quad \boxed{S = \frac{1}{(1 - x)(1 - 2x)} \quad (|x| < 1/2)}$$

AH Maths - MiA (2nd Edn.)

- pg. 175 Ex. 10.3 Q 1 a - g.

Feb 10-15:47

Ex. 10.3

1 For each of these power series,

i find the sum to infinity

ii identify the values of x for which the sum to infinity is valid.

a $S = 2 + 6x + 10x^2 + 14x^3 + \ldots + (4n - 2)x^n + \ldots$

b $S = 3 + 8x + 13x^2 + 18x^3 + \ldots + (5n - 2)x^n + \ldots$

c $S = 12 + 10x + 8x^2 + 6x^3 + \ldots + (14 - 2n)x^n + \ldots$

d $S = 17 + 12x + 7x^2 + 2x^3 + \ldots + (22 - 5n)x^n + \ldots$

e $S = 2 + 6x + 18x^2 + 54x^3 + \ldots$

f $S = 96 + 48x + 24x^2 + 12x^3 + \ldots$

g $S = 1 + 4x + 13x^2 + 40x^3 + \ldots$

Answers to AH Maths (MiA), pg. 175, Ex. 10.3

1	a	i	$\frac{2(1+x)}{(1-x)^2}$	ii	$-1 < x < 1$
	b	i	$\frac{3+2x}{(1-x)^2}$	ii	$-1 < x < 1$
	c	i	$\frac{12-14x}{(1-x)^2}$	ii	$-1 < x < 1$
	d	i	$\frac{17-22x}{(1-x)^2}$	ii	$-1 < x < 1$
	e	i	$\frac{2}{1-3x}$	ii	$-\frac{1}{3} < x < \frac{1}{3}$
	f	i	$\frac{192}{2-x}$	ii	$-2 < x < 2$
	g	i	$\frac{1}{(1-3x)(1-x)}$	ii	$-\frac{1}{3} < x < \frac{1}{3}$