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Unit 1 : Differential Calculus - Lesson 6

Higher-Order Derivatives

LI

- Work out Higher-Order Derivatives of a function.
- Use Higher-Order Derivatives to prove identities.

SC

- Repeated differentiation.

Higher-Order Derivatives

When a function $y = f(x)$ is differentiated once and that derivative is differentiated, the result is the
2nd Derivative of y :

$$\frac{d^2}{dx^2} y = \frac{d}{dx}\left(\frac{d}{dx} y\right)$$

The n^{th} Order-Derivative of y is the function obtained by differentiating y n times :

$$\frac{d^n}{dx^n} y = \underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} \cdots \left(\frac{d}{dx}\right.\right.\right.}_{\text{differentiate } n \text{ times}} y \left.\left.\left.\right) \cdots \right)\right)$$

$$y^{\overbrace{'' \cdots '}^{n \text{ dashes}}} \text{ or } y^{(n)}$$

$$D^n y$$

Example 1

Find the first 4 derivatives of $y = x^5$.

$$y = x^5$$

$$\therefore \quad \boxed{y' = 5x^4}$$

$$\Rightarrow \quad \boxed{y'' = 20x^3}$$

$$\Rightarrow \quad \boxed{y''' = 60x^2}$$

$$\Rightarrow \quad \boxed{y^{(4)} = 120x}$$

Example 2

Find the smallest value of n for which $f^{(n)}(x) = 0$ if $f(x) = x^4$.

$$f(x) = x^4$$

$$\therefore \quad f'(x) = 4x^3$$

$$\Rightarrow \quad f''(x) = 12x^2$$

$$\Rightarrow \quad f^{(3)}(x) = 24x$$

$$\Rightarrow \quad f^{(4)}(x) = 24$$

$$\Rightarrow \quad \underline{f^{(5)}(x) = 0}$$

As the 5^{th} derivative is 0, all other higher-order derivatives are also 0. So,

$$\boxed{\text{Smallest n-value : } n = 5}$$

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Example 3

Given that $y = e^x \cos x$, show that,

$$D^4 y = -4y$$

$$y = e^x \cos x$$

$$\therefore \quad Dy = (e^x)\cos x + e^x(-\sin x)$$

$$\Rightarrow \quad \underline{Dy = e^x(\cos x - \sin x)}$$

$$\therefore \quad D^2 y = (e^x)(\cos x - \sin x) + e^x(-\sin x - \cos x)$$

$$\Rightarrow \quad \underline{D^2 y = -2e^x \sin x}$$

$$\therefore \quad D^3 y = -2(e^x)\sin x - 2e^x(\cos x)$$

$$\Rightarrow \quad \underline{D^3 y = -2e^x(\sin x + \cos x)}$$

$$\therefore \quad D^4 y = -2(e^x)(\sin x + \cos x) - 2e^x(\cos x - \sin x)$$

$$\Rightarrow \quad D^4 y = -4e^x \cos x$$

$$\Rightarrow \quad \boxed{D^4 y = -4y}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 60-1 Ex. 4.10 Q 2, 3, 4 (i) 5, 7, 8, 9 a.

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Ex. 4.10

2 a Find i $\frac{d}{dx}(2x+1)^3$ ii $\frac{d^2}{dx^2}(2x+1)^3$ iii $\frac{d^3}{dx^3}(2x+1)^3$

b What is the lowest value of n for which $\frac{d^n}{dx^n}(2x+1)^3 = 0$?

3 Find the derivatives which do not equal zero for $(2x+3)^4$.

4 For each of these functions

i write its first, second and third derivative

a $\cos x$ b $\sin 2x$

c $\frac{1}{x}$ [Hint: $(-1)^n = 1$ when n is even; $(-1)^n = -1$ when n is odd.]

d $\ln x$ e e^{3x} f $\sqrt{x}$ g xe^x

5 a Given $y_1 = \tan x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b Given $y_2 = \ln(\cos x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

c Write a connection between the derivatives of $y_2 = \ln(\cos x)$ and $y_1 = \tan x$.

7 Given that $y = \frac{x}{2x+1}$

a show that $\frac{dy}{dx} = \frac{y^2}{x^2}$

b show that $\frac{d^2y}{dx^2} = -4\frac{y^3}{x^3}$

c show that $\frac{d^3y}{dx^3} = 24\frac{y^4}{x^4}$.

8 Given that $y = e^x \sin x$ show that $\frac{d^4y}{dx^4} = -4y$.

9 $y = \sqrt[3]{(x-1)^4}$

a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Answers to AH Maths (MiA), pg. 60-1, Ex. 4.10

2 a i $6(2x+1)^2$ ii $24(2x+1)$ iii 48

b 4

3 Multiplications not performed to emphasise pattern.

$8\,(2x+3)^3,\ 48\,(2x+3)^2,\ 192\,(2x+3),\ 384$

4 a i $-\sin x, -\cos x, \sin x$

b i $2\cos 2x,\ -4\sin 2x,\ -8\cos 2x$

c i $-x^{-2}, 2x^{-3}, -6x^{-4}$

d i $x^{-1}, -x^{-2}, 2x^{-3}$

e i $3e^{3x}, 9e^{3x}, 27e^{3x}$

f i $\frac{1}{2}x^{-\frac{1}{2}},\ -\frac{1}{4}x^{-\frac{3}{2}},\ \frac{3}{8}x^{-\frac{5}{2}}$

g i $e^x + xe^x, 2e^x + xe^x, 3e^x + xe^x$

5 a $\sec^2 x, 2\sec^2 x\tan x$ **b** $-\tan x, -\sec^2 x$

c $\frac{d^n y_1}{dx^n} = -\frac{d^{n+1} y_2}{dx^{n+1}}$

7 a $y = \frac{x}{2x+1} \Rightarrow \frac{y}{x} = \frac{1}{2x+1}$ and

$$\frac{dy}{dx} = \frac{(2x+1).1 - x.2}{(2x+1)^2} = \frac{1}{(2x+1)^2} = \frac{y^2}{x^2}$$

b $\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^3} = -4\frac{y^3}{x^3}$

c $\frac{d^3y}{dx^3} = \frac{24}{(2x+1)^4} = 24\frac{y^4}{x^4}$

8 $y = e^x \sin x \Rightarrow \frac{dy}{dx} = e^x\sin x + e^x\cos x$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\sin x + e^x\cos x + e^x\cos x - e^x\sin x$$
$$= 2e^x\cos x$$
$$\Rightarrow \frac{d^3y}{dx^3} = 2e^x\cos x - 2e^x\sin x$$
$$\Rightarrow \frac{d^4y}{dx^4} = 2e^x\cos x - 2e^x\sin x - 2e^x\sin x - 2e^x\cos x$$
$$= -4e^x\sin x = -4y$$

9 a $\frac{dy}{dx} = \frac{4}{3}(x-1)^{\frac{1}{3}}; \frac{d^2y}{dx^2} = \frac{4}{9}(x-1)^{-\frac{2}{3}}$

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