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Matrices and Systems of Equations - Lesson 6

Determinants and Inverses of 3 x 3 Matrices

LI

- Calculate 3 x 3 Determinants.
- Calculate the Inverse of a 3 x 3 matrix.
- Solve systems of equations using inverses.
- Matrix Properties 3.

SC

- Primary school arithmetic.

The determinant of $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is,

$$\det A = |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

where $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ is the determinant of $\begin{pmatrix} e & f \\ h & i \end{pmatrix}$ etc. .

A formula exists for the inverse of a 3 x 3 matrix, but it is very complicated; we instead use row reduction (as in Gaussian Elimination).

Finding the inverse of a 3×3 matrix A involves row-reducing the giant augmented matrix,

$$(A \mid I)$$

to the form,

$$(I \mid B)$$

Then $B = A^{-1}$.

Feb 2-09:46

The reason the preceding procedure works is explained thus :

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{p}} \quad \longleftrightarrow \quad (A \mid \underline{\mathbf{p}})$$

$$A^{-1}A\,\underline{\mathbf{x}} = A^{-1}\,\underline{\mathbf{p}}$$

$$\mathrm{I}\,\underline{\mathbf{x}} = A^{-1}\,\underline{\mathbf{p}} \quad \longleftrightarrow \quad (\mathrm{I} \mid A^{-1}\,\underline{\mathbf{p}})$$

Choosing the vector $\underline{\mathbf{p}}$ to be each of the three columns of the identity matrix, each $\underline{\mathbf{p}}$ picks out the columns of A^{-1}. The calculation is more convenient if we start with the whole identity matrix, instead of its separate columns.

Matrix Properties - 3

13) $|AB| = |A|\,|B|$

14) $|k\,A| = k^{n}\,|A|$, if A has order $n \times n$

15) $|A^{T}| = |A|$

16) $(AB)^{-1} = B^{-1}A^{-1}$

17) $(k\,A)^{-1} = (1/k)\,A^{-1}$

18) $(A^{T})^{-1} = (A^{-1})^{T}$

19) $|A^{-1}| = 1/|A|$

Jan 29-09:48

Example 1

Find k so that $A = \begin{pmatrix} 10 & k & 0 \\ k & 1 & 3 \\ -3 & 0 & 1 \end{pmatrix}$ is singular.

For singularity, we require $|A| = 0$. So,

$$10\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} - k\begin{vmatrix} k & 3 \\ -3 & 1 \end{vmatrix} + 0\begin{vmatrix} k & 1 \\ -3 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 10(1 - 0) - k(k + 9) = 0$$

$$\Rightarrow \quad 10 - k^2 - 9k = 0$$

$$\Rightarrow \quad k^2 + 9k - 10 = 0$$

$$\Rightarrow \quad (k + 10)(k - 1) = 0$$

$$\Rightarrow \quad \boxed{k = -10, k = 1}$$

Example 2

Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -3 & -2 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - (3/5)R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1/5 & -19/5 & -3/5 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to (-5)R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

$$\xrightarrow{R_2 \longrightarrow R_2 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & 0 & 55 & 10 & -15 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

$$\xrightarrow{R_2 \longrightarrow (-1/5) R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -11 & -2 & 3 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

$$\xrightarrow{R_1 \longrightarrow R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 12 & 2 & -3 \\ 0 & 1 & 0 & -11 & -2 & 3 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

$$\xrightarrow{R_1 \longrightarrow R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -1 & 2 \\ 0 & 1 & 0 & -11 & -2 & 3 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

$$\therefore \quad \boxed{A^{-1} = \begin{pmatrix} -7 & -1 & 2 \\ -11 & -2 & 3 \\ 19 & 3 & -5 \end{pmatrix}}$$

Example 3

Show that for an orthogonal matrix A, $|A| = \pm 1$.

If A is orthogonal,

$$AA^T = I$$

$$\therefore \quad |AA^T| = |I|$$

$$\Rightarrow \quad |A|\,|A^T| = |I|$$

$$\Rightarrow \quad |A|\,|A| = |I|$$

$$\Rightarrow \quad |A|^2 = 1$$

$$\Rightarrow \quad \boxed{|A| = \pm 1}$$

Sep 12-14:22

AH Maths - MiA (2nd Edn.)

- pg. 247-9 Ex. 13.9 Q 4, 5.
- pg. 275-6 Ex. 14.10
  Q 1 a, e, 2 d, g, h.

Ex. 13.9

4 Evaluate each determinant.

a $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 5 \\ 3 & 6 & 8 \end{vmatrix}$ b $\begin{vmatrix} -1 & 2 & 1 \\ 3 & 4 & 5 \\ 2 & -7 & 3 \end{vmatrix}$ c $\begin{vmatrix} 4 & 1 & 3 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$ d $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ -1 & -1 & -1 \end{vmatrix}$

5 Simplify each determinant.

a $\begin{vmatrix} a & b & c \\ 4 & 7 & -1 \\ 3 & -2 & 0 \end{vmatrix}$ b $\begin{vmatrix} p & 2 & 4 \\ q & -1 & 3 \\ r & 1 & -2 \end{vmatrix}$ c $\begin{vmatrix} 1 & k & 2 \\ 4 & m & -1 \\ -3 & n & 2 \end{vmatrix}$ d $\begin{vmatrix} 1 & 0 & 1 \\ 0 & x & 0 \\ 1 & 0 & 1 \end{vmatrix}$

Ex. 14.10

1 Using elementary row operations, find the inverse of each of these non-singular matrices.

a $\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ e $\begin{pmatrix} 5 & 10 & 2 \\ 1 & 5 & 2 \\ 4 & 3 & -1 \end{pmatrix}$

2 For each of these matrices A

i show that A is invertible [by showing $|A| \neq 0$]

ii find the inverse matrix, A^{-1}, by using elementary row operations.

d $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$ g $\begin{pmatrix} 3 & 2 & 1 \\ 5 & 6 & 4 \\ 7 & 2 & 3 \end{pmatrix}$ h $\begin{pmatrix} -1 & 4 & 1 \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix}$

Answers to AH Maths (MiA), pg. 247-9, Ex. 13.9

4 **a** -19 **b** -74 **c** 2 **d** 0

5 **a** $-(2a + 3b + 29c)$ **b** $10r + 8q - p$

c $8m - 5k + 9n$ **d** 0

Answers to AH Maths (MiA), pg. 275-6, Ex. 14.10

1 **a** $\begin{pmatrix} -1 & 2 & -2 \\ 3 & -6 & 7 \\ 1 & -1 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} -11 & 16 & 10 \\ 9 & -13 & -8 \\ -17 & 25 & 15 \end{pmatrix}$

2 **d** $\frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ -12 & 4 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ **g** $\frac{1}{24}\begin{pmatrix} 10 & -4 & 2 \\ 13 & 2 & -7 \\ -32 & 8 & 8 \end{pmatrix}$ **h** $\frac{1}{18}\begin{pmatrix} 1 & 6 & 7 \\ 3 & 0 & 3 \\ 7 & 6 & -5 \end{pmatrix}$