Integration - Lesson 6

Definite Integrals

LI

- Evaluate definite integrals.

SC

- Integrate.
- Substitution.

The definite integral of a function $y = f(x)$ between lower limit a and upper limit b is written as,

$$\int_a^b f(x)\ dx$$

For specific numbers a and b, the definite integral gives a real number.

If $w(x)$ is the result of integrating $f(x)$, then the above is written as,

$$\Big[\, w(x) \,\Big]_a^b$$

This is then evaluated as,

$$w(b) - w(a)$$

Example 1 (Non-Calc)

Find $\int_1^4 (4x^3 - 2x)\,dx$.

$$\int_1^4 (4x^3 - 2x)\,dx$$

$$= \left[x^4 - x^2\right]_1^4$$

$$= \left(4^4 - 4^2\right) - \left(1^4 - 1^2\right)$$

$$= (256 - 16) - (0)$$

$$= \boxed{240}$$

Example 2 (Non-Calc)

Find $\int_{-2}^{0} (5x + 1)(3x - 2)\, dx$.

$$\int_{-2}^{0} (5x + 1)(3x - 2)\, dx$$

$$= \int_{-2}^{0} (15x^2 + 3x - 10x - 2)\, dx$$

$$= \int_{-2}^{0} (15x^2 - 7x - 2)\, dx$$

$$= \left[5x^3 - \frac{7x^2}{2} - 2x \right]_{-2}^{0}$$

$$= (0) - \left(5(-2)^3 - \frac{7(-2)^2}{2} - 2(-2) \right)$$

$$= -(-40 - 14 + 4)$$

$$= \boxed{50}$$

Example 3 (Non-Calc)

Find $\int_{-1}^{1} (2x - 1)^3 \, dx$.

$$\int_{-1}^{1} (2x - 1)^3 \, dx$$

$$= \left[\frac{(2x - 1)^4}{4(2)} \right]_{-1}^{1}$$

$$= \frac{1}{8} \left[(2x - 1)^4 \right]_{-1}^{1}$$

$$= \frac{1}{8} \left((2(1) - 1)^4 - (2(-1) - 1)^4 \right)$$

$$= \frac{1}{8} (1 - 81)$$

$$= \boxed{-10}$$

Example 4 (Non-Calc)

Find $\int_{-2}^{-3/4} \sqrt{1 - 4x}\, dx$.

$$\int_{-2}^{-3/4} \sqrt{1 - 4x}\, dx$$

$$= \int_{-2}^{-3/4} (1 - 4x)^{1/2}\, dx$$

$$= \left[\frac{(1 - 4x)^{3/2}}{3/2\,(-4)} \right]_{-2}^{-3/4}$$

$$= -\frac{1}{6}\left[\left(\sqrt{1 - 4x}\right)^3 \right]_{-2}^{-3/4}$$

$$= -\frac{1}{6}\left(\left(\sqrt{1 + 3}\right)^3 - \left(\sqrt{1 + 8}\right)^3 \right)$$

$$= -\frac{1}{6}(8 - 27)$$

$$= \boxed{\frac{19}{6}}$$

Example 5 (Non-Calc)

Find $\int_0^{\pi/6} 8 \cos x \; dx$.

$$\int_0^{\pi/6} 8 \cos x \; dx$$

$$= \Big[8 \sin x \Big]_0^{\pi/6}$$

$$= 8 \Big[\sin x \Big]_0^{\pi/6}$$

$$= 8 (\sin (\pi/6) - \sin 0)$$

$$= 8 (1/2 - 0)$$

$$= \boxed{4}$$

Example 6 (Non-Calc)

Find $\int_{\pi/8}^{\pi/4} 4 \sin 4x \, dx$.

$$\int_{\pi/8}^{\pi/4} 4 \sin 4x \, dx$$

$$= \left[\frac{4(-\cos 4x)}{4} \right]_{\pi/8}^{\pi/4}$$

$$= -\left[\cos 4x \right]_{\pi/8}^{\pi/4}$$

$$= -(\cos \pi - \cos(\pi/2))$$

$$= -(-1 - 0)$$

$$= \boxed{1}$$

Example 7 (Calc)

Find $\int_2^4 \frac{1}{5} \cos(3x - 1)\, dx$ to 2 s. f..

$$\int_2^4 \frac{1}{5} \cos(3x - 1)\, dx$$

$$= \left[\frac{\sin(3x - 1)}{5(3)} \right]_2^4$$

$$= \frac{1}{15} \left[\sin(3x - 1) \right]_2^4$$

$$= \frac{1}{15} (\sin 11 - \sin 5)$$

$$= -0.002\,737\ldots$$

$$= \boxed{-0.002\,7 \ (2 \text{ s. f.})}$$

Example 8 (Calc)

Find $\int_{-3}^{\pi/12} (5 + \sin x)\, dx$ to 2 d. p. .

$$\int_{-3}^{\pi/12} (5 + \sin x)\, dx$$

$$= \Big[5x - \cos x \Big]_{-3}^{\pi/12}$$

$$= \big(5\pi/12 - \cos(\pi/12)\big) - \big(-15 - \cos(-3)\big)$$

$$= 14.353\ldots$$

$$= \boxed{14.35 \text{ (2 d. p.)}}$$

CfE Higher Maths

- pg. 286 - 7 Ex. 12A
  Q 1, 2, 3 a-e, 4 d-f, 5, 7, 9.

- pg. 289 - 290 Ex. 12B
  Q 2, 3 d-f, 4 - 6.

Questions

1 Evaluate these integrals.

a $\int_{1}^{3} 2x + 5 \, dx$ **b** $\int_{0}^{4} 6 - 2x \, dx$ **c** $\int_{-2}^{1} 3x^2 + 4x - 1 \, dx$

d $\int_{-1}^{2} 2x^4 \, dx$ **e** $\int_{-2}^{2} 4x^3 + x^2 - 3x + 2 \, dx$ **f** $\int_{0}^{6} 5 - 2x^2 \, dx$

2 Evaluate these integrals.

a $\int_{1}^{4} (x + 2)(3x + 2) \, dx$ **b** $\int_{-2}^{1} x(2x - 1)^2 \, dx$ **c** $\int_{-1}^{2} (x + 1)\left(x^2 - 4\right) \, dx$

d $\int_{2}^{4} (x + 1)\left(x^2 + 2x - 3\right) \, dx$ **e** $\int_{-3}^{3} 2x^2(1 - x) - 3\left(x^2 - 2\right) \, dx$

3 Evaluate these integrals.

a $\int_{1}^{3} \frac{4}{x^2} \, dx$ **b** $\int_{3}^{4} \frac{1}{2x^3} \, dx$ **c** $\int_{-2}^{-1} \frac{4}{3x^5} - 1 \, dx$

d $\int_{4}^{9} 3\sqrt{x} \, dx$ **e** $\int_{1}^{4} \frac{2}{\sqrt{x}} \, dx$

4 Evaluate these integrals.

d $\int_{-3}^{-1} (2x + 3)^5 \, dx$ **e** $\int_{0}^{1} (5x - 4)^3 \, dx$ **f** $\int_{1}^{2} \frac{1}{8}(3x - 5)^8 \, dx$

5 Evaluate these integrals.

a $\int_{1}^{3} \frac{x^3 - 2}{x^2}\, dx$

b $\int_{4}^{9} \sqrt{x}\left(2 - \frac{1}{x}\right) dx$

c $\int_{-2}^{-1} \frac{1}{(1-x)^3}\, dx$

d $\int_{1}^{6} \sqrt{x+3}\, dx$

e $\int_{4}^{16} \frac{2}{x^3} - \frac{3}{4}\sqrt{x}\, dx$

f $\int_{6}^{8} \left(5 - \frac{1}{2}x\right)^3 dx$

g $\int_{-3}^{-1} 6 - \frac{1}{2}x - \frac{1}{3x^2}\, dx$

h $\int_{\frac{1}{2}}^{\frac{5}{3}} 8(x-1)^3\, dx$

i $\int_{\sqrt{2}}^{\sqrt{3}} x\left(x^2 - 1\right) dx$

7 Determine algebraically the values of k such that $\int_{-2}^{k} 4x - 3\, dx = -15$.

9 Determine algebraically the negative value of p for which $\int_{0}^{3} \frac{7}{3(2x+p)^2}\, dx = 1$.

Answers

1 **a** 18
b 8
c 0
d 13.2
e 40/3
f −114

2 **a** 135
b −28.5
c −11.25
d 104
e 18

3 **a** 8/3
b 7/576
c −1.3125
d 38
e 4

4 **d** − 182/3
e −12.75
f 2.375

5 **a** 8/3
b 70/3
c 5/72
d 38/3
e − 7 153/256
f 7.5
g 124/9
h 175/648
i 0.75

7 $k = 0.5$
$k = 1$

9 −7

Questions

2 Evaluate these integrals.

a $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\cos\left(x - \frac{\pi}{6}\right)\, dx$ **b** $\int_{0}^{\frac{\pi}{4}} 6\sin\left(x + \frac{\pi}{4}\right)\, dx$ **c** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos x\, dx$

d $\int_{0}^{\frac{\pi}{6}} 3\cos 2x\, dx$ **e** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x\, dx$ **f** $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos\frac{1}{2}x\, dx$

3 Evaluate these integrals.

d $\int_{\frac{\pi}{12}}^{\pi} 4\sin(3x - \pi)\, dx$ **e** $\int_{-\frac{5\pi}{3}}^{-\frac{\pi}{3}} \cos\frac{1}{2}x\, dx$ **f** $\int_{-\pi}^{\pi} \sin\frac{2}{3}x\, dx$

4 Evaluate these integrals, giving your answers to 2 decimal places.

a $\int_{1}^{5} 2\sin x\, dx$ **b** $\int_{2.3}^{3.6} -4\cos x\, dx$ **c** $\int_{2}^{4} \sin(4x - 1)\, dx$

d $\int_{-1.2}^{3.4} \frac{2}{3}\cos(2 - x)\, dx$ **e** $\int_{3}^{5} \cos\left(2x - \frac{\pi}{2}\right) + 3\, dx$ **f** $\int_{-1.3}^{0.5} 4 - \sin(5 - 2x)\, dx$

5 Find the values of t, $0 \le t < 2\pi$, for which $\int_{\frac{\pi}{6}}^{t} 3\cos x\, dx = \frac{3\sqrt{2}}{2} - \frac{3}{2}$.

6 Find the values of p, $0 \le p < \pi$, for which $\int_{0}^{p} \sin 2x\, dx = \frac{1}{4}$.

Answers

2 **a** $\frac{3}{2}$

b $3\sqrt{2}$

c 4

d $\frac{3\sqrt{3}}{4}$

e $\frac{1}{3}$

f $\sqrt{3} - 1$

3 **d** $\frac{-2\left(2+\sqrt{2}\right)}{3}$

e 0

f 0

4 **a** 0.51

b 4.75

c 0.38

d 0.62

e 6.90

f 7.65

5 $t = \frac{\pi}{4}$

$t = \frac{3\pi}{4}$

6 $p = \frac{\pi}{6}$

$p = \frac{5\pi}{6}$