Integration - Lesson 5

Solving Differential Equations

LI

- Solve equations of the form $\frac{dy}{dx} = f(x)$.

SC

- Integrate.
- Solve simple equations.

A differential equation is an equation containing derivatives.

In Higher Maths, we will study differential equations that can be written in the form :

$$\frac{dy}{dx} = f(x)$$

A differential equation is to be solved for the function y.

The function y is solved for by integrating both sides of the above differential equation.

Remembering that integration and differentiation are opposite processes, differentiating a function y and then integrating it gives the same function back, generally with a non-zero constant.

$$\int \frac{dy}{dx}\, dx = \int f(x)\, dx$$

Integrating $f(x)$ gives a new function $w(x)$.

$$\therefore \quad y(x) + C_1 = w(x) + C_2$$

$$\Rightarrow \quad \boxed{y(x) = w(x) + C}$$

where $C = C_2 - C_1$.

The boxed equation above for y is the general solution of the differential equation.

The integration constant C can be worked out if a value for x and a corresponding value for y are given. Then the solution is a particular solution (aka integral curve) of the differential equation.

Example 1

Solve $\frac{dy}{dx} = x^2$ for y.

With no further information, we are clearly looking for the general solution of this differential equation.

$$\frac{dy}{dx} = x^2$$

$$\therefore \quad y = \int x^2 \, dx$$

$$\Rightarrow \quad \boxed{y = \frac{x^3}{3} + C}$$

Example 2

For a given function y it is known that

$\frac{dy}{dx} = x^2 - 6$ and $x = -3$ when $y = 7$.

Express y in terms of x.

We first find the general solution by integrating; then we use the x and y values to find C, thus obtaining the particular solution for this pair of x and y values.

$$\frac{dy}{dx} = x^2 - 6$$

$$\therefore \quad y = \int x^2 - 6 \; dx$$

$$\Rightarrow \quad \underline{y = \frac{x^3}{3} - 6x + C}$$

Using the information that $x = -3$ when $y = 7$, we have,

$$7 = \frac{(-3)^3}{3} - 6(-3) + C$$

$$\Rightarrow \quad 7 = -9 + 18 + C$$

$$\Rightarrow \quad \underline{C = -2}$$

$$\therefore \quad \boxed{y = \frac{x^3}{3} - 6x - 2}$$

Example 3

A curve has equation $y = f(x)$. Given that

$f'(x) = \dfrac{2}{\sqrt{x}}$ and $f(16) = 80$, find the equation

of the curve.

$$f'(x) = \frac{2}{\sqrt{x}}$$

$$f'(x) = 2x^{-1/2}$$

$$\therefore \quad f(x) = \int 2x^{-1/2}\,dx$$

$$\Rightarrow \quad f(x) = \frac{2x^{1/2}}{1/2} + C$$

$$\Rightarrow \quad f(x) = 4x^{1/2} + C$$

$$\Rightarrow \quad \underline{f(x) = 4\sqrt{x} + C}$$

Using $f(16) = 80$,

$$80 = 4\sqrt{16} + C$$

$$\Rightarrow \quad 80 = 16 + C$$

$$\Rightarrow \quad \underline{C = 64}$$

$$\therefore \quad \boxed{f(x) = 4\sqrt{x} + 64}$$

Example 4

A curve has equation $y = g(x)$. Given that $g'(x) = 6\cos 3x$ and the point $(\pi/2, 3)$ lies on the curve, express $g(x)$ in terms of x.

$$g'(x) = 6\cos 3x$$

$$\therefore \quad g(x) = \int 6\cos 3x \, dx$$

$$\Rightarrow \quad \underline{g(x) = 2\sin 3x + C}$$

Using $x = \pi/2$ and $y = g(x) = 3$,

$$3 = 2\sin(3\pi/2) + C$$

x must be in radians

$$\Rightarrow \quad 3 = 2(-1) + C$$

$$\Rightarrow \quad \underline{C = 5}$$

$$\therefore \quad \boxed{y(x) = 2\sin 3x + 5}$$

Higher Maths for CfE

pg. 209 Ex. 15 . 9 All Q

Questions

1 Find the equations of the curves which satisfy the following conditions:

a) $\frac{dy}{dx} = 6x$ and the curve passes through the point $(2, 5)$.

b) $\frac{dy}{dx} = x^2 - 2x + 1$ and the curve passes through the point $(3, 8)$.

c) $\frac{dy}{dx} = \sqrt{4x+1}$ and the curve passes through the point $(2, 6)$.

d) $\frac{dy}{dx} = \cos 2x$ and the curve passes through the point $\left(\frac{\pi}{12}, 1\right)$.

2 Find the solutions of these differential equations.

a) $\frac{ds}{dt} = 3t^2 + 4t + 5$ if $s = 1$ when $t = -1$.

b) $\frac{ds}{dt} = 2t - \frac{20}{t^2}$ if $s = 13$ when $t = -4$.

c) $\frac{dv}{dt} = \sqrt[3]{t}$ if $v = 5$ when $t = 8$.

d) $\frac{dv}{dt} = 3\sin(4t - \pi)$ if $v = 1$ when $t = \frac{\pi}{2}$.

3 The gradient of the tangent to a curve at the point (x, y) is given by $\frac{dy}{dx} = \frac{9+2x^3}{x^2}$. HPQ
If the curve passes through the point $(-3, 10)$, find its equation.

4 The rate of change of a function is given by $f'(x) = (2x-3)^3$.
Find $f(x)$ given that $f(2{\cdot}5) = 4$.

5 The rate of change of a function is given by $g'(\theta) = 2\cos 3\theta$.
Find $g(\theta)$ given that $g\left(\frac{\pi}{6}\right) = 2$.

6 The speed of a roller-coaster is given by $d'(t) = 20t - 5$, where t is the time in seconds after the start of the ride.
Find $d(t)$ given that 3 seconds after the start of the ride the roller-coaster had travelled 75 metres. ACE

Answers

1 **a)** $y = 3x^2 - 7$

b) $y = \frac{1}{3}x^3 - x^2 + x + 5$

c) $y = \frac{1}{6}\sqrt{(4x+1)^3} + \frac{3}{2}$

d) $y = \frac{1}{2}\sin 2x + \frac{3}{4}$

2 **a)** $s = t^3 + 2t^2 + 5t + 5$

b) $s = t^2 + \frac{20}{t} + 2$

c) $v = \frac{3}{4}\sqrt[3]{t^4} - 7$

d) $v = \frac{1}{4}(1 - 3\cos(4t - \pi))$

3 $y = x^2 - \frac{9}{x} - 2$

4 $f(x) = \frac{(2x-3)^4}{8} + 2$

5 $g(\theta) = \frac{2}{3}(\sin 3\theta + 2)$

6 $d(t) = 10t^2 - 5t$

MiA Higher Maths

pg. 126 Ex. 1 All Q

Questions

1 Find the general solutions of these differential equations.

a $\frac{dy}{dx} = 2x$ **b** $\frac{dy}{dx} = 4x$ **c** $\frac{dy}{dx} = 6x + 1$

Check each answer by differentiation.

2 Find the particular solutions of these differential equations for the given values of x and y.

a $\frac{dy}{dx} = 3 + 6x$, given that $y = 2$ when $x = -1$

b $\frac{dy}{dx} = 12x + 1$, given that $y = 16$ when $x = 2$

c $\frac{dy}{dx} = 5 - 8x$, given that $y = -2$ when $x = 1$.

3 The table gives the gradients of the tangents at (x, y) to six curves, and a point on each curve. Find the equations of the curves.

	$\frac{dy}{dx}$	Curve passes through the point
a	$6x$	$(1, 5)$
b	$8x$	$(-2, 4)$
c	$2x - 1$	$(2, 1)$
d	$4x - 3$	$(1, 0)$
e	$1 - 3x^2$	$(2, 2)$
f	$4x^3$	$(-1, -1)$

4 Kate and Mike make a simultaneous parachute jump. Their velocity after x seconds is $v = 5 + 10x$ m/s. If they have fallen y metres, then $v = \frac{dy}{dx}$, so $\frac{dy}{dx} = 5 + 10x$.

a Find the distance y metres they fall in x seconds, given $y = 0$ when $x = 0$.

b Calculate the distance they fall in 10 seconds.

Answers

1 **a** $y = x^2 + c$ **b** $y = 2x^2 + c$
c $y = 3x^2 + x + c$

2 **a** $y = 3x + 3x^2 + 2$ **b** $y = 6x^2 + x - 10$
c $y = 5x - 4x^2 - 3$

3 **a** $y = 3x^2 + 2$ **b** $y = 4x^2 - 12$
c $y = x^2 - x - 1$ **d** $y = 2x^2 - 3x + 1$
e $y = x - x^3 + 8$ **f** $y = x^4 - 2$

4 **a** $y = 5x + 5x^2$ **b** 550 m