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Unit 1 : Integral Calculus - Lesson 5

Reduction Formulae

LI

- Obtain a recursive formula for calculating similar integrals.

SC

- Integration by Parts formula.
- Recurrence relations.

A recursive formula (aka recurrence relation) is one that allows values to be obtained from previous ones

A reduction formula is a recursive formula involving integrals

In this lesson, 'n' will always represent a whole number and 'a' will always represent a real number

Example 1

Obtain a reduction formula for $\int \cos^n x \, dx$.

Hence obtain an expression for $\int \cos^5 x \, dx$.

$$\text{Let } I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

$$u = \cos^{n-1} x, \quad v = \sin x$$

$$u' = -(n-1) \sin x \cos^{n-2} x, \quad v' = \cos x$$

In the following, we make the useful abbreviations

$S = \sin x$ and $C = \cos x$; so, for example,

$$\sin^2 x = S^2 \text{ and } \cos^n x = C^n.$$

$$I_n = uv - \int u'v \, dx$$

$$\therefore I_n = SC^{n-1} + (n-1) \int S^2 C^{n-2} \, dx$$

$$\Rightarrow I_n = SC^{n-1} + (n-1) \int (1 - C^2) C^{n-2} \, dx$$

$$\Rightarrow I_n = SC^{n-1} + (n-1) \int (C^{n-2} - C^n) \, dx$$

$$\Rightarrow I_n = SC^{n-1} + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n = SC^{n-1} + (n-1) I_{n-2} - n I_n + I_n$$

$$\Rightarrow n I_n = SC^{n-1} + (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n} SC^{n-1} + \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

This reduction formula allows us to work out I_5 if we know I_3 ; I_3 can be worked out if we know I_1 .

$$I_1 = \int \cos x \, dx \Rightarrow I_1 = \underline{\sin x}$$

$$I_3 = \frac{1}{3} \sin x \cos^{3-1} x + \frac{3-1}{3} I_{3-2}$$

$$\Rightarrow I_3 = \frac{1}{3} \sin x \cos^2 x + \frac{2}{3} I_1$$

$$\Rightarrow I_3 = \frac{1}{3} \sin x \cos^2 x + \frac{2}{3} \sin x$$

$$I_5 = \frac{1}{5} \sin x \cos^{5-1} x + \frac{5-1}{5} I_{5-2}$$

$$\Rightarrow I_5 = \frac{1}{5} \sin x \cos^4 x + \frac{4}{5} I_3$$

$$\Rightarrow I_5 = \frac{1}{5} \sin x \cos^4 x + \frac{4}{5} \left(\frac{1}{3} \sin x \cos^2 x + \frac{2}{3} \sin x \right)$$

$$I_5 = \frac{1}{5} \sin x \cos^4 x + \frac{4}{15} \sin x \cos^2 x + \frac{8}{15} \sin x + C$$

Example 2

Find a reduction formula for $\int_0^1 x^n e^{-3x} dx$.

Hence obtain $\int_0^1 x^2 e^{-3x} dx$ as an exact value.

Let $I_n = \int_0^1 x^n e^{-3x} dx$.

$$\boxed{u = x^n, \quad v = -(1/3)e^{-3x}}$$

$$\boxed{u' = nx^{n-1}, \quad v' = e^{-3x}}$$

$$I_n = [uv]_a^b - \int_a^b u'v dx$$

$$\therefore I_n = -\frac{1}{3} [x^n e^{-3x}]_0^1 + \frac{n}{3} \int_0^1 x^{n-1} e^{-3x} dx$$

$$\Rightarrow I_n = -\frac{1}{3} (1 \cdot e^{-3} - 0) + \frac{n}{3} \int_0^1 x^{n-1} e^{-3x} dx$$

$$\Rightarrow I_n = -\frac{1}{3} e^{-3} + \frac{n}{3} I_{n-1}$$

$$I_0 = \int_0^1 e^{-3x} dx \Rightarrow I_0 = -\frac{1}{3} e^{-3} + \frac{1}{3}$$

$$I_1 = -\frac{1}{3} e^{-3} + \frac{1}{3} I_0$$

$$\Rightarrow I_1 = -\frac{1}{3} e^{-3} + \frac{1}{3} \left(-\frac{1}{3} e^{-3} + \frac{1}{3} \right)$$

$$\Rightarrow I_1 = -\frac{1}{3} e^{-3} - \frac{1}{9} e^{-3} + \frac{1}{9}$$

$$\Rightarrow I_1 = -\frac{4}{9} e^{-3} + \frac{1}{9}$$

$$I_2 = -\frac{1}{3} e^{-3} + \frac{2}{3} I_1$$

$$\Rightarrow I_2 = -\frac{1}{3} e^{-3} + \frac{2}{3} \left(-\frac{4}{9} e^{-3} + \frac{1}{9} \right)$$

$$\Rightarrow I_2 = -\frac{1}{3} e^{-3} - \frac{8}{27} e^{-3} + \frac{2}{27}$$

$$\Rightarrow I_2 = \frac{2}{27} - \frac{17}{27} e^{-3}$$

Questions

1) Obtain a reduction formula for $\int \sin^n x \, dx$.

Hence obtain an expression for $\int \sin^5 x \, dx$.

2) Find a reduction formula for $\int_0^1 x^n e^{2x} \, dx$.

Hence obtain $\int_0^1 x^2 e^{2x} \, dx$ as an exact value.

Answers

$$1) \quad I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$$

$$I_5 = -\frac{1}{5} \cos x \sin^4 x - \frac{4}{15} \cos x \sin^2 x - \frac{8}{15} \cos x + C$$

$$2) \quad I_n = \frac{1}{2} e^2 - \frac{n}{2} I_{n-1}$$

$$I_2 = \frac{1}{4} e^2 - \frac{1}{4}$$