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Unit 2 : Sequences and Series - Lesson 5

Infinite Series, Partial Sums and Infinite Geometric Series

LI

- Know what a Sum to Infinity is.
- Determine if a sum to infinity exists for an infinite geometric series.
- Calculate the sum to infinity for an infinite geometric series.
- Solve problems involving finite geometric series.

SC

- Sum to infinity formula.

An infinite series is obtained by adding infinitely many terms of a sequence

The partial sum of a sequence is the sum to n terms S_n

The sum to infinity (aka infinite sum) of a sequence is all the terms of the sequence added together; it is also given by the limit as $n \to \infty$ of the partial sums :

$$S_\infty = \lim_{n \to \infty} S_n$$

S_∞ for an Arithmetic Series

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Expanding brackets and rearranging gives S_n in the form,

$$S_n = Pn^2 + Qn$$

As $n \longrightarrow \infty$, $S_n \longrightarrow \pm\infty$. So,

$\boxed{S_\infty \text{ does not exist for an arithmetic series}}$

S_∞ for a Geometric Series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$r > 1$ or $r < -1$:

As $n \to \infty$, $r^n \to \pm\infty$. So, $S_n \to \pm\infty$. Hence, S_∞ does not exist.

$-1 < r < 1$:

As $n \to \infty$, $r^n \to 0$. So, $S_n \to \dfrac{a(1 - 0)}{1 - r} = \dfrac{a}{1 - r}$.

For a geometric series, S_∞ exists provided that $-1 < r < 1$ (i.e. $|r| < 1$):

$$S_\infty = \frac{a}{1 - r}$$

Example 1

Show that the sum to infinity for the geometric series
$36 + 12 + 4 + \ldots$ exists and hence find this sum.

We have $a = 36$ and $r = 4/12 \Rightarrow r = 1/3$.

$$\boxed{\text{As } -1 < 1/3 < 1,\ S_\infty \text{ exists}}$$

$$S_\infty = \frac{a}{1 - r}$$

$$\therefore \quad S_\infty = \frac{36}{1 - (1/3)}$$

$$\Rightarrow \quad S_\infty = \frac{36}{2/3}$$

$$\therefore \quad \boxed{S_\infty = 54}$$

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Example 2

Find the condition on x for the series $1 + 3x + 9x^2 + \ldots$ to have a sum to infinity.

$$9x^2 \div 3x = 3x$$

$$3x \div 1 = 3x$$

A successive ratios are constant, the series is geometric. We also have $a = 1$ and $r = 3x$. S_∞ will exist if $|r| < 1$, i.e. when,

$$|3x| < 1$$

$$\Rightarrow \quad 3|x| < 1$$

$$\Rightarrow \quad \underline{|x| < 1/3}$$

$$\boxed{S_\infty \text{ will exist if } |x| < \frac{1}{3}, \text{ i.e. } -\frac{1}{3} < r < \frac{1}{3}}$$

Example 3

Write $0.363\,636\ldots$ as a vulgar fraction in simplest form.

$$0.363\,636\ldots = 0.36 + 0.0036 + 0.000036 + \ldots$$

$$\Rightarrow \quad 0.363\,636\ldots = \frac{36}{10^2} + \frac{36}{10^4} + \frac{36}{10^6} + \ldots$$

$$\Rightarrow \quad 0.363\,636\ldots = 36\,(10^{-2} + 10^{-4} + 10^{-6} + \ldots)$$

The bracketed term is a geometric series with common ratio 10^{-2} and first term 10^{-2}. As $-1 < 10^{-2} < 1$, the sum to infinity exists. So,

$$0.363\,636\ldots = 36\left(\frac{10^{-2}}{1 - 10^{-2}}\right)$$

$$\Rightarrow \quad 0.363\,636\ldots = 36\left(\frac{1}{10^2 - 1}\right)$$

$$\Rightarrow \quad 0.363\,636\ldots = \frac{36}{99}$$

$$\Rightarrow \quad \boxed{0.363\,636\ldots = \frac{12}{33}}$$

Example 4

Write $\frac{1}{1 - 2x}$ in the form $\sum_{r=0}^{\infty} k^r x^r$, stating the value of the real number k and the range of values of x for which the sum is valid.

$\frac{1}{1 - 2x}$ is of the form of the infinite sum $\frac{a}{1 - r}$, which is the limit of a geometric sequence with common ratio r and first term a (we have $a = 1$ and $r = 2x$), provided $|r| < 1$. The expansion will be valid provided that $|2x| < 1 \Rightarrow 2|x| < 1 \Rightarrow |x| < 1/2$. So, if this is the case,

$$\frac{1}{1 - 2x} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots$$

$$\Rightarrow \frac{1}{1 - 2x} = (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots$$

$$\Rightarrow \frac{1}{1 - 2x} = \sum_{r=0}^{\infty} (2x)^r$$

$$\Rightarrow \boxed{\frac{1}{1 - 2x} = \sum_{r=0}^{\infty} 2^r x^r \quad \left(k = 2, |x| < \frac{1}{2}\right)}$$

Example 5

Write $\dfrac{1}{5 + x}$ in the form $\sum_{r=0}^{\infty} (-1)^r k^{r+1} x^r$, stating the value of the real number k and the range of values of x for which the sum is valid.

$$\frac{1}{5 + x} = \frac{1}{5(1 + (x/5))} = \frac{1}{5}\left(\frac{1}{1 - (-x/5)}\right)$$

$\dfrac{1}{1 - (-x/5)}$ is of the form of the infinite sum $\dfrac{a}{1 - r}$, which is the limit of a geometric sequence with common ratio r and first term a (we have $a = 1$ and $r = -x/5$), provided $|r| < 1$. The expansion will be valid provided that $|-x/5| < 1$ $\Rightarrow |x/5| < 1 \Rightarrow (1/5)|x| < 1 \Rightarrow |x| < 5$; then,

$$\frac{1}{5 + x} = (1/5)(1 + (-x/5) + (-x/5)^2 + (-x/5)^3 + \ldots)$$

$$\Rightarrow \frac{1}{5 + x} = (1/5)\sum_{r=0}^{\infty} (-x/5)^r$$

$$\Rightarrow \frac{1}{5 + x} = (1/5)\sum_{r=0}^{\infty} (-1)^r (1/5)^r x^r$$

$$\Rightarrow \boxed{\frac{1}{5 + x} = (1/5)\sum_{r=0}^{\infty} (-1)^r (1/5)^{r+1} x^r \quad \left(k = \frac{1}{5},\ |x| < 5\right)}$$

AH Maths - MiA (2nd Edn.)

- pg. 162-4 Ex. 9.5 Q 1, 2 a - c, 3 a, 4 a, b, 5, 7 a - f.

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Ex. 9.5

1 Find the sum to infinity of these infinite geometric progressions.

a $2 + 1 + 0.5 + \dots$ b $100 + 20 + 4 + \dots$ c $8 + 2 + 0.5 + \dots$

d $9 - 3 + 1 - \dots$ e $500 - 100 + 20 - \dots$ f $147 + 21 + 3 + \dots$

2 Identify which of these geometric series tend to a limit and find the limit.

a $7 + 1 + \frac{1}{7} + \dots$ b $1 + 1.5 + 2.25 + \dots$ c $18 + 9 + 4.5 + \dots$

3 a A geometric series has a sum to infinity of 50.
The common ratio is 0.6.
What is the first term of the series?

4 a Express each of these recurring decimals as an infinite geometric series, and hence as a vulgar fraction in its simplest form.

i 0.282828... ii 0.345345... iii 0.041414...

b By considering a sum to infinity, show that 0.99999... is mathematically equivalent to 1.

5 Given that numbers 75 and 18.75 are adjacent terms of an infinite geometric series with a sum to infinity of 6400, find

a the first term

b the partial sum S_5.

7 Identify for what range of values of x these series have a sum to infinity.

a $3 + 3x + 3x^2 + \dots$ b $16 + 8x + 4x^2 + \dots$

c $\frac{1}{x} + 4 + 16x + \dots, x \neq 0$ d $5 + 20\sqrt{x} + 80x + \dots$

e $2 + \frac{6}{x} + \frac{18}{x^2} + \dots$ f $32x^{10} + 8x^8 + 2x^6 + \dots$

Answers to AH Maths (MiA), pg. 162-4, Ex. 9.5

1 **a** 4 **b** 125 **c** $10\frac{2}{3}$

d $6\frac{3}{4}$ **e** $416\frac{2}{3}$ **f** $171\frac{1}{2}$

2 **a** $8\frac{1}{6}$ **b** No limit **c** 36

3 **a** 20

4 **a** **i** $\frac{28}{99}$ **ii** $\frac{115}{333}$ **iii** $\frac{41}{990}$

b $0{\cdot}9 + 0{\cdot}09 + 0{\cdot}009 + \ldots\ a = 0{\cdot}9, r = 0{\cdot}1 \Rightarrow S_\infty = 1$

5 **a** 4800 **b** $6393{\cdot}75$

7 **a** $-1 < x < 1$ **b** $-2 < x < 2$

c $-\frac{1}{4} < x < \frac{1}{4}$ **d** $0 < x < \frac{1}{16}$

e $x < -3$ or $x > 3$ **f** $x < -\frac{1}{2}$ or $x > \frac{1}{2}$

1) Write $\dfrac{1}{1 - 7x}$ in the form $\displaystyle\sum_{r=0}^{\infty} k^r x^r$, stating the value of the real number k and the range of values of x for which the sum is valid.

2) Write $\dfrac{1}{6 + x}$ in the form $\displaystyle\sum_{r=0}^{\infty} (-1)^r k^{r+1} x^r$, stating the value of the real number k and the range of values of x for which the sum is valid.

3) Write $\dfrac{4}{9 + x}$ in the form $\displaystyle\sum_{r=0}^{\infty} p(-1)^r k^{r+1} x^r$, stating the value of the real numbers p and k and the range of values of x for which the sum is valid.

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1) $\dfrac{1}{1 - 7x} = \sum_{r=0}^{\infty} 7^r x^r \quad \left(k = 7, |x| < \dfrac{1}{7}\right)$

2) $\dfrac{1}{6 + x} = \sum_{r=0}^{\infty} (-1)^r (1/6)^{r+1} x^r \quad \left(k = \dfrac{1}{6}, |x| < 6\right)$

3) $\dfrac{4}{9 + x} = \sum_{r=0}^{\infty} 4(-1)^r 9^{-(r+1)} x^r \quad \left(p = 4, k = 9, |x| < 9\right)$

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