Differential Calculus - Lesson 5

Derivatives of Sin x and Cos x

LI

- Know the derivatives of $\sin x$ and $\cos x$.
- Use these derivatives to solve problems involving tangents.

SC

- Differentiation rules for $\sin x$ and $\cos x$.
- Exact values of $\sin x$ and $\cos x$ for $x = 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$.
- Graphs of $\sin x$ and $\cos x$ in radians.

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

These equations are only true when x is in RADIANS

If x is in degrees, the equations are more complicated; that's why we use radians when doing calculus with trigonometric functions

Example 1

Differentiate $y = 5\cos x$ with respect to x.

$$y = 5\cos x$$

$$\therefore \quad \boxed{y' = -5\sin x}$$

Example 2

Find the derivative of $y = \dfrac{3 \sin x - 4 \cos x}{7}$.

$$y = \frac{3 \sin x - 4 \cos x}{7}$$

$$y = \frac{3}{7} \sin x - \frac{4}{7} \cos x$$

$$\therefore \quad \boxed{y' = \frac{3}{7} \cos x + \frac{4}{7} \sin x}$$

Example 3

Find the gradient of the tangent to the curve

$y = 2\cos x$ at $x = \frac{\pi}{4}$.

$$y(x) = 2\cos x$$

$$\therefore \quad y'(x) = -2\sin x$$

$$\therefore \quad y'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad y'\left(\frac{\pi}{4}\right) = -2\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \boxed{y'\left(\frac{\pi}{4}\right) = -\sqrt{2}}$$

Example 4

Find the rate of change of $p(y) = 4\sin y$ at

at $y = \dfrac{4\pi}{3}$.

$$p(y) = 4\sin y$$

$$\therefore \quad p'(y) = 4\cos y$$

$$\therefore \quad p'\left(\frac{4\pi}{3}\right) = 4\cos\left(\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad p'\left(\frac{4\pi}{3}\right) = 4 \,.\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad \boxed{p'\left(\frac{4\pi}{3}\right) = -2}$$

Example 5

Find the gradient of the tangent to the curve with equation $y = 3\cos x - 4x$ when $x = 2$.

Give your answer correct to 2 s.f..

$$y(x) = 3\cos x - 4x$$

$$\therefore \quad y'(x) = -3\sin x - 4$$

$$\therefore \quad y'(2) = -3\sin 2 - 4$$

$$\Rightarrow \quad y'(2) = -3(0.909\ldots) - 4$$

$$\Rightarrow \quad \boxed{y'(2) = -6.73 \text{ (to 2 s.f.)}}$$

Example 6

A curve has equation $y = 2\sin x \ (0 \le x \le \pi)$.

Find the x - coordinates of the points where the gradient of the curve is $-\sqrt{3}$.

$$y(x) = 2\sin x$$

$$\therefore \quad \underline{y'(x) = 2\cos x}$$

$y'(x) = -\sqrt{3}$ implies,

$$2\cos x = -\sqrt{3}$$

$$\Rightarrow \quad \cos x = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{RAA} = \frac{\pi}{6}$$

$$\Rightarrow \quad \underline{x = \frac{5\pi}{6}, \frac{7\pi}{6}}$$

But the range of x - values is $0 \le x \le \pi$. So,

$$\boxed{x = \frac{5\pi}{6}}$$

Example 7

Find the equation of the tangent to $y = 6 \sin x$

at $x = \frac{\pi}{6}$.

$$y(x) = 6 \sin x$$

$$\therefore \quad y'(x) = 6 \cos x$$

$$\therefore \quad y'\left(\frac{\pi}{6}\right) = 6 \cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad y'\left(\frac{\pi}{6}\right) = 6 \, . \, \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \underline{y'\left(\frac{\pi}{6}\right) = 3\sqrt{3}}$$

$$y(x) = 6 \sin x$$

$$\therefore \quad y\left(\frac{\pi}{6}\right) = 6 \sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad y\left(\frac{\pi}{6}\right) = 6 \, . \, \frac{1}{2}$$

$$\Rightarrow \quad \underline{y\left(\frac{\pi}{6}\right) = 3}$$

$$y - b = m(x - a)$$

$$\boxed{\begin{array}{c} m = 3\sqrt{3} \\ (a, b) \\ \frac{\pi}{6} \quad 3 \end{array}}$$

$$\therefore \quad y - 3 = 3\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$\Rightarrow \quad y - 3 = 3\sqrt{3}\,x - \frac{\pi\sqrt{3}}{2}$$

$$\Rightarrow \quad 2y - 6 = 6\sqrt{3}\,x - \pi\sqrt{3}$$

$$\Rightarrow \quad \boxed{2y = 6\sqrt{3}\,x - \pi\sqrt{3} + 6}$$

CfE Higher Maths

- pg. 227 - 228 Ex. 9E Q 1 - 7
- pg. 241 Ex. 10A Q 2 a, c, 4 a, c

Questions

1 Differentiate:

a $8\sin x$ **b** $3\cos x$ **c** $-\sin x$

d $\frac{1}{2}\sin x$ **e** $\frac{2}{3}\cos x$ **f** $-\frac{5}{8}\cos x$

g $6x^2 + 7\sin x$ **h** $3\sin x + 7\cos x$ **i** $\cos x + 6\sin x$

j $\sin x - \cos x$ **k** $\frac{3}{x^2} - \cos x$ **l** $\frac{4}{5}\sin x - 6\sqrt{x}$

m $-5\cos x + \frac{3}{4x}$ **n** $5x^3 - \frac{1}{\sqrt[3]{x^5}} + 9\sin x$ **o** $\frac{1 - 9x^2\cos x}{3x^2}$

p $\frac{4 - x^2}{x^3} - \frac{1}{5}\cos x$ **q** $\frac{6}{\sqrt{x}} - 4\sin x$ **r** $-\frac{5}{6}\cos x - \frac{5 - \sqrt{x}}{x^2}$

s $\frac{\sin x - 3\cos x}{5}$

2 **a** Given $f(x) = 6\sin x$, find the value of $f'\left(\frac{\pi}{3}\right)$.

b $y = 2\cos x$ Find $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$.

c Find the gradient of the tangent to the curve with equation $y = \frac{1}{2}\sin x$ at the point where $x = \frac{\pi}{4}$.

d On a suitable domain, the function f is defined by $f(x) = -4\cos x$. Find the rate of change of f when $x = \frac{\pi}{3}$.

3 **a** A curve has equation $y = 4\sin x$. Find the gradient of the curve at the point where $x = \frac{5\pi}{3}$.

b On a suitable domain, the function g is defined by $g(x) = 5\cos x$. Find $g'\left(\frac{7\pi}{6}\right)$.

c Find the gradient of the tangent to the curve $y = \frac{3}{4}\sin x$ at the point where $x = \frac{2\pi}{3}$.

d $y = -6\sin x$ Find $\frac{dy}{dx}$ when $x = \frac{5\pi}{4}$.

4 A function is defined by $y = 4\sin x - \cos x$. Find $\frac{dy}{dx}$ when $x = \frac{3\pi}{2}$.

5 **You may use a calculator in this question.** Give your answers to 2 decimal places.

a Find the gradient of the tangent to the curve with equation $y = 3\sin x - 4$ when $x = 2$.

b On a suitable domain, the function f is defined by $f(x) = -4\cos x + 1$. Evaluate $f'(4)$.

c Find the gradient of the curve with equation $y = 7\sin x - 5\cos x$ at the point where $x = 2.2$.

d On a suitable domain the function g is defined by $4x - \frac{1}{2}\sin x$. Find the rate of change of g when $x = -1.6$.

6 The function f is defined by $f(x) = 6\sin x$ where $0 \le x < 2\pi$. Find the values of x for which the rate of change of f is equal to 3.

7 A curve has equation $y = 8\cos x$. Find the values of x, $0 \le x < 2\pi$, for which $\frac{dy}{dx} = -4$

Answers

1
- **a** $8\cos x$
- **b** $-3\sin x$
- **c** $-\cos x$
- **d** $\frac{1}{2}\cos x$
- **e** $-\frac{2}{3}\sin x$
- **f** $\frac{5}{8}\sin x$
- **g** $12x + 7\cos x$
- **h** $3\cos x - 7\sin x$
- **i** $6\cos x - \sin x$
- **j** $\cos x + \sin x$
- **k** $\sin x - \frac{6}{x^3}$
- **l** $\frac{4\cos x}{5} - \frac{3}{\sqrt{x}}$
- **m** $5\sin x - \frac{3}{4x^2}$
- **n** $9\cos x + 15x^2 + \frac{5}{3x^{\frac{8}{3}}}$
- **o** $3\sin x - \frac{2}{3x^3}$
- **p** $-\frac{12}{x^4} + \frac{1}{x^2} + \frac{1}{5}\sin x$
- **q** $-4\cos x - \frac{3}{x^{\frac{3}{2}}}$
- **r** $\frac{5}{6}\sin x + \frac{10}{x^3} - \frac{3}{2x^{\frac{5}{2}}}$
- **s** $\frac{1}{5}(\cos x + 3\sin x)$

2
- **a** 3
- **b** -1
- **c** $\frac{1}{2\sqrt{2}}$
- **d** $2\sqrt{3}$

3
- **a** 2
- **b** $\frac{5}{2}$
- **c** $-\frac{3}{8}$
- **d** $3\sqrt{2}$

4 -1

5
- **a** -1.25
- **b** -3.03
- **c** -0.08
- **d** 4.01

6 $x = \frac{\pi}{3}$

$x = \frac{5\pi}{3}$

7 $x = \frac{\pi}{6}$

$x = \frac{5\pi}{6}$

Questions

2 For each of these functions, find the equation of the tangent at the given point.

a $y = \cos x;\ \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

c $y = 4\sin x;\ \left(\frac{\pi}{4}, 2\sqrt{2}\right)$

4 For each of these functions, find the equation of the tangent at the given point.

a $y = \cos x;\ x = \frac{\pi}{3}$

c $y = 8\cos x;\ x = \frac{2\pi}{3}$

Answers

2 **a** $y = -\frac{1}{2}x + \frac{1}{12}\left(\pi + 6\sqrt{3}\right)$

c $y = 2\sqrt{2}x + 2\sqrt{2}\left(1 - \frac{\pi}{4}\right)$

4 **a** $y = -\frac{\sqrt{3}}{2}x + \frac{1}{6}\left(3 + \sqrt{3}\pi\right)$

c $y = -4\sqrt{3}x + \frac{4}{3}\left(2\sqrt{3}\pi - 3\right)$