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Unit 1 : Differential Calculus - Lesson 5

Derivatives of Inverse (Trigonometric) Functions

LI

- Know how to differentiate the inverse of a function.
- Know the derivatives of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

SC

- Chain Rule.

Jan 10-12:11

Derivative of an Inverse Function

If a function $y = f(x)$ has inverse f^{-1}, then,

$$f(f^{-1}(x)) = x$$

$$\therefore \quad \frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x$$

$$\Rightarrow \quad f'(f^{-1}(x)) \, . \, \frac{d}{dx} f^{-1}(x) = 1$$

$$\boxed{' \text{ means } \frac{d}{dx}}$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}}$$

(Derivative of an inverse function)

This result is sometimes written,

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

Example 1

Find the derivative of the inverse of the function $f(x) = x^3$.

$$f(x) = x^3 \quad \begin{cases} f'(x) = 3x^2 \\ f^{-1}(x) = x^{1/3} \end{cases}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\therefore \quad \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(x^{1/3})}$$

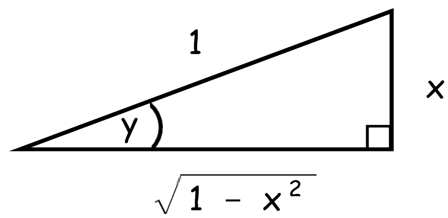
$$\Rightarrow \quad \frac{d}{dx} f^{-1}(x) = \frac{1}{3(x^{1/3})^2}$$

$$\Rightarrow \quad \boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{3x^{2/3}}}$$

Derivatives of Inverse Trigonometric Functions

Let $x = \sin y$. Hence, $y = \sin^{-1} x$. Later, we will see that the inverse only makes sense for acute angles y.

$$\sin y = x = \frac{x}{1}$$



$$\cos y = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}$$

$$x = \sin y \Rightarrow \frac{dx}{dy} = \cos y$$

$$\frac{d}{dx} \sin^{-1} x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

- $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}$

Major notational warning : $\sin^{-1} x$ does not mean $\dfrac{1}{\sin x}$

$\sin^{-1}$	is pronounced	'inverse sine' or 'arcsine' (asin)
$\cos^{-1}$	''	'inverse cosine' or 'arccosine' (acos)
$\tan^{-1}$	''	'inverse tangent' or 'arctangent' (atan)

Jun 9-07:31

General Form of Derivatives - Chain Rule

$$\frac{d}{dx}\sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1 - (f(x))^2}}$$

$$\frac{d}{dx}\cos^{-1} f(x) = -\frac{f'(x)}{\sqrt{1 - (f(x))^2}}$$

$$\frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1 + (f(x))^2}$$

Jun 8-10:14

Example 2

Differentiate $\sin^{-1}(5x)$ with respect to x.

$$g(x) = \sin^{-1}(5x)$$

$$\therefore \quad g'(x) = \frac{1}{\sqrt{1 - (5x)^2}} \cdot \frac{d}{dx}(5x)$$

$$\Rightarrow \quad \boxed{g'(x) = \frac{5}{\sqrt{1 - 25x^2}}}$$

Example 3

If $f(x) = \tan^{-1}(e^{4x})$, find the exact value of $f'(1/2)$.

$$f(x) = \tan^{-1}(e^{4x})$$

$$\therefore \quad f'(x) = \frac{1}{1 + (e^{4x})^2} \cdot \frac{d}{dx}(e^{4x})$$

$$\Rightarrow \quad f'(x) = \frac{4e^{4x}}{1 + e^{8x}}$$

$$\therefore \quad f'(1/2) = \frac{4e^{4(1/2)}}{1 + e^{8(1/2)}}$$

$$\Rightarrow \quad \boxed{f'(1/2) = \frac{4e^{2}}{1 + e^{4}}}$$

Example 4

Find the gradient of the tangent line to the curve defined by $y = x\cos^{-1} x$ at $x = 0$.

$$y(x) = x\cos^{-1} x$$

This is a product of functions of x, so the Product Rule must be used.

$$\therefore \quad y'(x) = (1)\cos^{-1} x + x \cdot \frac{-1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad y'(x) = \cos^{-1} x - \frac{x}{\sqrt{1 - x^2}}$$

$$\therefore \quad y'(0) = \cos^{-1}(0) - \frac{0}{\sqrt{1 - 0^2}}$$

$$\Rightarrow \quad y'(0) = \pi/2 - 0$$

$$\Rightarrow \quad \boxed{y'(0) = \pi/2}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 84 Ex. 6.1 Q 1 a, b, d.
- pg. 85 Ex. 6.2 Q 1 a, b, d, f, 2 a, b, d, e, 3 a, c, d, 4 - 6, 8.
- pg. 86-7 Ex. 6.3 Q 1 - 7, 9 a.

Feb 10-15:47

Ex. 6.1

1 For each function $f(x)$, find the derivative of the inverse function $f^{-1}(x)$.

- a $f(x) = x^5$
- b $f(x) = x^{\frac{3}{4}}$
- c $f(x) = 2x^{-2}, x > 0$
- d $f(x) = x^2 + 1, x > 1$

Ex. 6.2

1 Find the derivative of

- a $\sin^{-1} x^2$
- b $\tan^{-1}(x+2)$
- c $\sin^{-1}\frac{1}{x}$
- d $\tan^{-1}\frac{1}{\sqrt{x}}$
- e $\cos^{-1}\frac{1}{x}$
- f $\cos^{-1} ax$

2 Find the derived function in each case.

- a $\sin^{-1}(e^x)$
- b $\cos^{-1}(x+2)^2$
- c $\sin^{-1}\sqrt{1-x^2}$
- d $\sin^{-1}(\tan x)$
- e $\sin^{-1}\left(\frac{x}{a}\right)$

3 Find $f'(x)$ for each of these expressions for $f(x)$.

- a $\cos^{-1} e^{2x}$
- b $\sin^{-1}\cos(x-1)$
- c $\tan^{-1}(1+x)$
- d $\cos^{-1}(\ln 3x)$
- e $\sec^{-1} 3x$

4 Differentiate

- a $\ln(\tan^{-1}\sqrt{x})$
- b $\ln\left(\sin^{-1}\frac{1}{\sqrt{x}}\right)$
- c $\ln(\sin^{-1} e^x)$
- d $e^{\sin^{-1} x}$

5 Calculate

- a $f'(1)$ where $f(x) = e^{\tan^{-1}\frac{1}{x}}$
- b $f'(0)$ where $f(x) = \ln(\cos^{-1} x)$
- c $f'\left(\frac{3}{4}\right)$ where $f(x) = \sin(\tan^{-1} x)$
- d $f'(\sqrt{3})$ where $f(x) = \cos\left(\tan^{-1}\frac{1}{x}\right)$

6 Find the gradient of the curve with equation

- a $y = \ln(\sin^{-1} 2x)$ where $x = \frac{1}{4}$
- b $y = (x + \tan^{-1} x)^3$ where $x = 1$
- c $y = \ln(\cos^{-1}(1-x))$ where $x = \frac{1}{2}$
- d $y = e^{\tan^{-1} x^2}$ where $x = 1$

8 Show that $y = (\sin^{-1} 3x)^4$ has a minimum turning point at the origin.

Ex. 6.3

1 Differentiate

a $x^2 \sin^{-1} x$ b $x \sin^{-1} x^2$ c $\sqrt{x} \cos^{-1} x$ d $\sqrt{x} \sin^{-1} \sqrt{x}$

2 Find the derivative of

a $(1 + x^2) \tan^{-1} x$ b $e^x \sin^{-1} x$ c $e^{2x} \cos^{-1}\left(\frac{x}{2}\right)$ d $\ln x \tan^{-1} x$

3 Find the derived function for each of these.

a $f(x) = \dfrac{\tan^{-1} x}{x}$ b $f(x) = \dfrac{\sin^{-1} x}{\sqrt{x}}$ c $f(x) = \dfrac{\cos^{-1} 2x}{x\sqrt{x}}$ d $f(x) = \dfrac{\tan^{-1}(x + 1)}{x^2}$

4 Find $f'(x)$ for each of these.

a $f(x) = \dfrac{x}{\sin^{-1} x}$ b $f(x) = \dfrac{x^2}{\cos^{-1}(x - 1)}$

c $f(x) = \dfrac{e^x}{\sin^{-1} 2x}$ d $f(x) = \dfrac{\ln x}{\tan^{-1} x}$

5 Calculate the gradient of the tangent to the curve with equation

a $y = \tan^{-1}\left(\dfrac{2x + 3}{3x - 2}\right)$ at $x = -\frac{1}{2}$ b $y = \sin^{-1}\left(\dfrac{1 + 2\cos x}{2 + \cos x}\right)$ at $x = \frac{\pi}{6}$

6 Find the coordinates of the stationary point on the curve with equation

a $y = \tan^{-1}\left(\dfrac{e^x}{x}\right)$ b $y = \cos^{-1}\left(\dfrac{\ln x}{x}\right)$

7 Show that there are no turning points on the curve with equation $y = \sin^{-1}\left(\dfrac{1 - x}{1 + x}\right)$.

9 a Show that $f(x) = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$ and $g(x) = 2\tan^{-1} x$ have the same derived function.

Answers to AH Maths (MiA), pg. 84, Ex. 6.1

1 a $\frac{1}{5}x^{-\frac{4}{5}}$ b $\frac{4}{3}x^{\frac{1}{3}}$ d $\dfrac{1}{2\sqrt{x-1}}$

Answers to AH Maths (MiA), pg. 85, Ex. 6.2

1 a $\dfrac{2x}{\sqrt{1-x^4}}$ b $\dfrac{1}{x^2+4x+5}$

d $\dfrac{-1}{2(x+1)\sqrt{x}}$

f $\dfrac{-a}{\sqrt{1-a^2x^2}}$

2 a $\dfrac{e^x}{\sqrt{1-e^{2x}}}$ b $\dfrac{-2(x+2)}{\sqrt{1-(x+2)^4}}$

d $\dfrac{1}{\cos x\sqrt{\cos 2x}}$

e $\dfrac{1}{\sqrt{a^2-x^2}}$

3 a $\dfrac{-2e^{2x}}{\sqrt{1-e^{4x}}}$

c $\dfrac{1}{x^2+2x+2}$ d $\dfrac{-1}{x\sqrt{1-(\ln 3x)^2}}$

4 a $\dfrac{1}{2(1+x)\sqrt{x}\tan^{-1}\sqrt{x}}$ b $\dfrac{-1}{2x\sqrt{x-1}\sin^{-1}\left(\frac{1}{\sqrt{x}}\right)}$

c $\dfrac{e^x}{\sin^{-1}(e^x)\sqrt{1-e^{2x}}}$ d $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

5 a $-\dfrac{e^{\frac{\pi}{4}}}{2}$ b $-\dfrac{2}{\pi}$

c $\dfrac{64}{125}$ d $\dfrac{1}{8}$

6 a $\dfrac{8\sqrt{3}}{\pi}$ b $\dfrac{9(4+\pi)^2}{32}$

c $\dfrac{2\sqrt{3}}{\pi}$ d $e^{\frac{\pi}{4}}$

8 $\dfrac{dy}{dx}=\dfrac{12(\sin^{-1}3x)^3}{\sqrt{1-9x^2}}$: SP when $12(\sin^{-1}3x)^3=0 : x=0$ is a solution. $y_{x=0}=0$

A table of signs confirms that it is a minimum TP.

Jun 1-11:51

Answers to AH Maths (MiA), pg. 86-7, Ex. 6.3

1 a $\dfrac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1}x$ b $\dfrac{2x^2}{\sqrt{1-x^4}} + \sin^{-1}x^2$

c $\dfrac{1}{2\sqrt{x}}\cos^{-1}x - \sqrt{\dfrac{x}{1-x^2}}$ d $\dfrac{1}{2\sqrt{1-x}} + \dfrac{\sin^{-1}\sqrt{x}}{2\sqrt{x}}$

2 a $1 + 2x\tan^{-1}x$ b $e^x\left[\sin^{-1}x + \dfrac{1}{\sqrt{1-x^2}}\right]$

c $e^{2x}\left[2\cos^{-1}\left(\dfrac{x}{2}\right) - \dfrac{1}{\sqrt{4-x^2}}\right]$

d $\dfrac{\ln x}{x^2+1} + \dfrac{\tan^{-1}x}{x}$

3 a $\dfrac{x-(1+x^2)\tan^{-1}x}{x^2(1+x^2)}$

b $\dfrac{2x-\sqrt{1-x^2}\sin^{-1}x}{2x\sqrt{x(1-x^2)}}$

c $\dfrac{4x+3\sqrt{1-4x^2}\cos^{-1}2x}{-2x^{\frac{5}{2}}\sqrt{1-4x^2}}$

d $\dfrac{x-2(x^2+2x+2)\tan^{-1}(x+1)}{x^3(x^2+2x+2)}$

4 a $\dfrac{\sqrt{1-x^2}\sin^{-1}x - x}{\sqrt{1-x^2}(\sin^{-1}x)^2}$

b $\dfrac{2x\sqrt{2x-x^2}\cos^{-1}(x-1)+x^2}{\sqrt{2x-x^2}(\cos^{-1}(x-1))^2}$

c $\dfrac{e^x\sqrt{1-4x^2}\sin^{-1}2x - 2e^x}{\sqrt{1-4x^2}(\sin^{-1}2x)^2}$ d $\dfrac{(1+x^2)\tan^{-1}x - x\ln x}{x(1+x^2)(\tan^{-1}x)^2}$

5 a $-\frac{4}{5}$ b $\dfrac{6-8\sqrt{3}}{13}$

6 a $(1, \tan^{-1}e)$ b $\left(e, \cos^{-1}\left(\frac{1}{e}\right)\right)$

7 $\dfrac{dy}{dx} = \dfrac{-1}{(1+x)\sqrt{x}} \neq 0$

9 a $f'(x) = g'(x) = \dfrac{2}{1+x^2}$