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Linear and Parabolic Motion - Lesson 5

Closest Approach and Collision Problems

LI

- Determine whether or not 2 objects collide and the time for collision.
- Find the closest distance between 2 moving objects and the time taken for this.

SC

- Vector Calculus.

(• Trigonometry.)

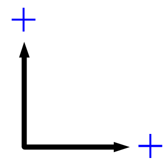
Collision Problems

- Equate displacement vectors

Unique solution for t implies collision; otherwise, no collision.

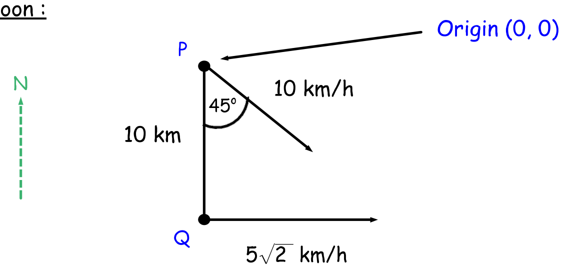
Example 1

At noon two ships P and Q are 10 kilometres apart with Q due south of P. P is sailing south east at a constant speed of 10 kmh^{-1} and Q is sailing due east at $5\sqrt{2}$ kmh^{-1}. Show that, if neither ship changes its velocity, they will collide and find the time, to the nearest minute, when the collision occurs.



Choose 12 noon to be when $t = 0$. Also, choose the origin of the coordinate system to be where P is when $t = 0$.

At noon :



P

$\underline{v}_P(t) = (10 \sin 45°)\,\underline{i} + (-10 \cos 45°)\,\underline{j}$

$\underline{v}_P(t) = 5\sqrt{2}\,\underline{i} - 5\sqrt{2}\,\underline{j}$

$\underline{r}_P(t) = (5\sqrt{2}\,t)\,\underline{i} - (5\sqrt{2}\,t)\,\underline{j} + \underline{C}$

$\underline{r}_P(0) = \underline{0}$ gives $\underline{C} = \underline{0}$.

$\underline{r}_P(t) = (5\sqrt{2}\,t)\,\underline{i} - (5\sqrt{2}\,t)\,\underline{j}$

Q

$\underline{v}_Q(t) = 5\sqrt{2}\,\underline{i}$

$\underline{r}_Q(t) = (5\sqrt{2}\,t)\,\underline{i} + \underline{D}$

$\underline{r}_Q(0) = -10\,\underline{j}$ gives
$\underline{D} = -10\,\underline{j}$.

$\underline{r}_Q(t) = (5\sqrt{2}\,t)\,\underline{i} - 10\,\underline{j}$

For a collision, there must be a t-value satisfying $\underline{r}_P(t) = \underline{r}_Q(t)$:

$$\underline{r}_P(t) = \underline{r}_Q(t)$$

$$\therefore \quad (5\sqrt{2}\,t)\,\underline{i} - (5\sqrt{2}\,t)\,\underline{j} = (5\sqrt{2}\,t)\,\underline{i} - 10\,\underline{j}$$

Equating components gives the y-component equation as the only non-trivial equation :

$$5\sqrt{2}\,t = 10$$

$$\Rightarrow \quad \underline{t = \sqrt{2}}$$

$\boxed{\text{As there is a single } t\text{-value satisfying } \mathbf{r}_P(t) = \mathbf{r}_Q(t)\text{, a collision occurs}}$

$$t = 1.41\ldots$$

$$\Rightarrow \quad \underline{t = 1\text{ h } 24.8\ldots\text{ min}}$$

$\boxed{\text{Time of collision is } 13\ 25}$

Useful result :

Solving $\frac{d}{dt} f = 0$ is the same as solving $\frac{d}{dt} \sqrt{f} = 0$.

Proof :

Let f be a function of the variable t.

$$\frac{d}{dt} \sqrt{f(t)}$$

$$= \frac{d}{dt} (f(t))^{1/2}$$

$$= (1/2 (f(t))^{-1/2} \frac{d}{dt} f(t)$$

i. e. $$\frac{d}{dt} \sqrt{f(t)} = \frac{\frac{d}{dt} f(t)}{(1/2 (f(t))^{1/2}}$$

Hence, if $\frac{d}{dt} \sqrt{f(t)} = 0$, then $\frac{d}{dt} f(t) = 0$,

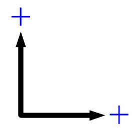
and vice versa.

Closest Approach

- Minimise distance between objects

Example 2

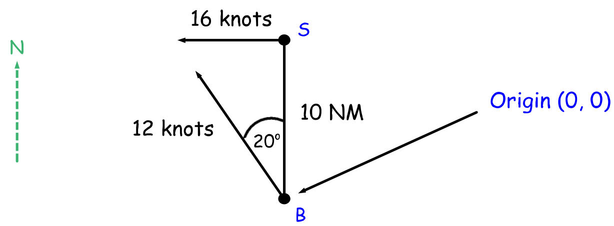
At noon a ship is 10 nautical miles due north of a boat. The ship is sailing due west at 16 knots and the boat is sailing at 12 knots on a bearing of 340°. Find the time when they are closest together and the distance apart at this time.



S : Ship
B : Boat

Choose 12 noon to be when $t = 0$. Also, choose the origin of the coordinate system to be where B is when $t = 0$.

At noon :



B	S
$\mathbf{v}_B(t) = (-12 \sin 20°)\,\mathbf{i} + (12 \cos 20°)\,\mathbf{j}$	$\mathbf{v}_S(t) = -16\,\mathbf{i}$
$\mathbf{r}_B(t) = (-12\,t \sin 20°)\,\mathbf{i} + (12\,t \cos 20°)\,\mathbf{j} + \mathbf{C}$	$\mathbf{r}_S(t) = (-16\,t)\,\mathbf{i} + \mathbf{D}$
$\mathbf{r}_B(0) = \mathbf{0}$ gives $\mathbf{C} = \mathbf{0}$.	$\mathbf{r}_S(0) = 10\,\mathbf{j}$ gives $\mathbf{D} = 10\,\mathbf{j}$.
$\mathbf{r}_B(t) = (-12\,t \sin 20°)\,\mathbf{i} + (12\,t \cos 20°)\,\mathbf{j}$	$\mathbf{r}_S(t) = (-16\,t)\,\mathbf{i} + 10\,\mathbf{j}$

Let $D(t)$ be the distance between B and S at time t. Then,

$$(D(t))^2 = (\mathbf{r}_S(t) - \mathbf{r}_B(t)) \bullet (\mathbf{r}_S(t) - \mathbf{r}_B(t))$$

$$\mathbf{r}_S(t) - \mathbf{r}_B(t) = (-11.89\ldots t)\,\mathbf{i} + (10 - 11.27\,t)\,\mathbf{j}$$

$$(D(t))^2 = 268.66\ldots t^2 - 225.52\ldots t + 100$$

$$\frac{d}{dt}(D(t))^2 = 0$$

$$537.32\ldots t - 225.52\ldots = 0$$

$$t = 0.419\ldots \text{h}$$

$$t = 25.18\ldots \text{min}$$

$\boxed{\text{Time of closest approach is 12 25 pm}}$

Let T_C be the time of closest approach, i. e. let $T_C = 0.419\ldots$; then,

$$D(T_C) = \sqrt{268.66\ldots T_C^{\,2} - 225.52\ldots T_C + 100}$$

$$D(T_C) = 7.257\ldots$$

$\boxed{\text{Closest distance is 7.3 nautical miles}}$

Blue Book

- pg. 236-239 Ex. 10 C Q 7.
- pg. 239-240 Ex. 10 D Q 1, 2.
- pg. 242-245 Ex. 10 E Q 2, 3, 11, 12, 14.
- pg. 245-248 Ex. 10 F Q 1, 4, 6.