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Geometry of Complex Numbers - Lesson 4

n^{th} Roots of a Complex Number

LI

- Find the n^{th} roots of a complex number.
- Find the n^{th} roots of unity.

SC

- Polar form.

If $w = r(\cos\theta + i\sin\theta)$, then the solutions of $z^n = w$ (the n^{th} roots of w) are,

$$z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

$(k = 0, 1, 2, 3, \ldots, n - 1)$

The roots z_k all lie on a circle with radius $r^{1/n}$ and any two successive solutions are spaced $2\pi/n$ radians apart

A special case occurs when $w = 1$, called the n^{th} roots of unity :

$$z_k = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

$(k = 0, 1, 2, 3, \ldots, n - 1)$

Example 1

Find the cube roots of i and plot them on an Argand diagram.

$$z^3 = i = 1\,\text{cis}\,(\pi/2)$$

$$\therefore\quad z_k = 1^{1/3}\,\text{cis}\,((\pi/2 + 2k\pi)/3)$$

$$\Rightarrow\quad z_k = \text{cis}\,((4k + 1)\pi/6)$$

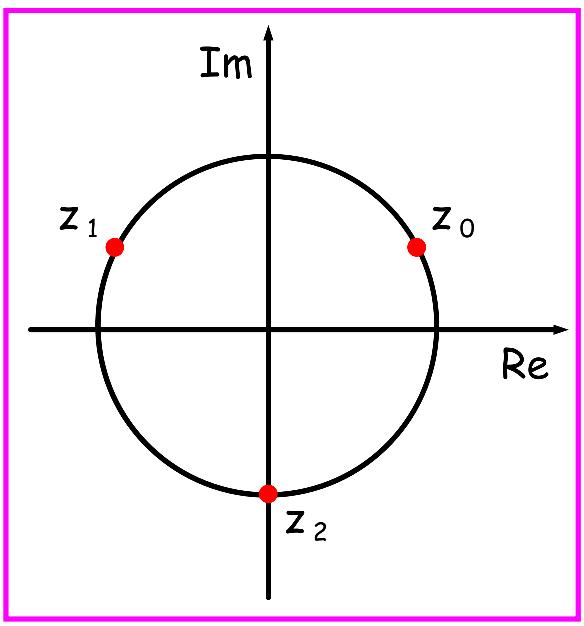
$$\Rightarrow\quad z_k = \cos((4k + 1)\pi/6) + i\sin((4k + 1)\pi/6)$$

$$(k = 0, 1, 2)$$

$$\therefore\quad z_0 = \cos(\pi/6) + i\sin(\pi/6) \Rightarrow \boxed{z_0 = \sqrt{3}/2 + i/2}$$

$$z_1 = \cos(5\pi/6) + i\sin(5\pi/6) \Rightarrow \boxed{z_1 = -\sqrt{3}/2 + i/2}$$

$$z_2 = \cos(3\pi/2) + i\sin(3\pi/2) \Rightarrow \boxed{z_2 = -i}$$



Example 2

Find the fourth roots of unity and plot them on an Argand diagram.

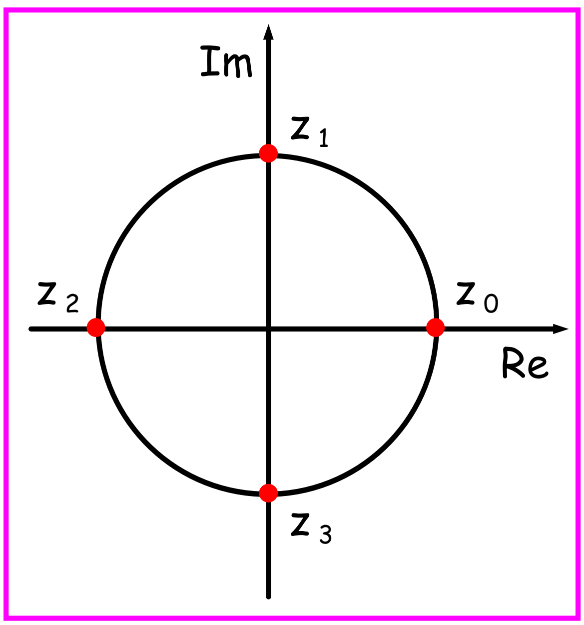
$$z^4 = 1 = 1\,\text{cis}\,(0)$$

$$\therefore \quad z_k = 1^{1/4}\,\text{cis}\,(2k\pi/4)$$

$$\Rightarrow \quad z_k = \text{cis}\,(k\pi/2)$$

$$\Rightarrow \quad \underline{z_k = \cos(k\pi/2) + i\sin(k\pi/2) \quad (k = 0, 1, 2, 3)}$$

$$\therefore \quad \boxed{z_0 = 1, z_1 = i, z_2 = -1, z_3 = -i}$$



Example 3

Solve $z^3 = 4 + 4\sqrt{3}\, i$.

Writing z^3 in polar form gives (check !),

$$z^3 = 8 \text{ cis } (\pi/3)$$

$$\therefore \quad z_k = 8^{1/3} \text{ cis } ((\pi/3 + 2k\pi)/3)$$

$$\Rightarrow \quad \underline{z_k = 2 \text{ cis } ((6k + 1)\pi/9) \quad (k = 0, 1, 2)}$$

$$\therefore \quad z_0 = 2 \text{ cis } (\pi/9),\ z_1 = 2 \text{ cis } (7\pi/9),\ z_2 = 2 \text{ cis } (13\pi/9)$$

$$\Rightarrow \quad \boxed{z_0 = 2 \text{ cis } (\pi/9),\ z_1 = 2 \text{ cis } (7\pi/9),\ z_2 = 2 \text{ cis } (-5\pi/9)}$$

Oct 2-23:10

AH Maths - MiA (2^{nd} Edn.)

- pg. 222 Ex. 12.7
  Q 1 (i) a - j, 2 a, c, d, e, f.

Feb 10-15:47

Ex. 12.7

1 For each of these

i solve the equation, leaving your answers in polar form.

a $z^3 = 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

b $z^4 = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

c $z^5 = 32\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$

d $z^3 = 64\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

e $z^5 = 32\left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right)$

f $z^3 = 64\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

g $z^4 = -2 - 2i$

h $z^5 = -3 + 3\sqrt{3}\,i$

i $z^4 = -2 + 2i$

j $z^5 = -3 - 3\sqrt{3}\,i$

2 a Solve $z^3 = 1$ to find the cube roots of unity. [Hint: $z^3 = 1(\cos 0 + i\sin 0)$]

c Find the sixth roots of unity.

d Solve the equation $z^4 = 81$.

e Find the complex numbers which satisfy the equation

i $z^5 = -1$ ii $z^5 = i$ iii $z^5 = -i$

f Find in polar form the solutions of

i $z^3 = -64$ ii $z^4 = 625i$ iii $z^5 = -\frac{i}{32}$

Answers to AH Maths (MiA), pg. 222, Ex. 12.7

1 **a** $2\text{ cis}\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right), k = 0, 1, 2$

b $\text{cis}\left(\frac{\pi}{20} + \frac{2k\pi}{4}\right), k = 0, 1, 2, 3$

c $2\text{ cis}\left(\frac{\pi}{35} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

d $4\text{ cis}\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right), k = 0, 1, 2$

e $2\text{ cis}\left(-\frac{\pi}{35} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

f $4\text{ cis}\left(-\frac{2\pi}{9} + \frac{2k\pi}{3}\right), k = 0, 1, 2$

g $2^{\frac{3}{8}}\text{ cis}\left(-\frac{3\pi}{16} + \frac{2k\pi}{4}\right), k = 0, 1, 2, 3$

h $6^{\frac{1}{5}}\text{ cis}\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

i $8^{\frac{1}{8}}\text{ cis}\left(\frac{3\pi}{16} + \frac{2k\pi}{4}\right), k = 0, 1, 2, 3$

j $6^{\frac{1}{5}}\text{ cis}\left(-\frac{2\pi}{15} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

2 **a** $\text{cis}\left(\frac{2k\pi}{3}\right), k = 0, 1, 2$

c $\text{cis}\left(\frac{2k\pi}{6}\right), k = 0, 1, 2, 3, 4, 5$

d $3\text{ cis}\left(\frac{2k\pi}{4}\right), k = 0, 1, 2, 3$

e **i** $\text{cis}\left(\frac{\pi + 2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

ii $\text{cis}\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

iii $\text{cis}\left(-\frac{\pi}{10} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

f **i** $4\text{ cis}\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right), k = 0, 1, 2$

ii $5\text{ cis}\left(\frac{\pi}{8} + \frac{2k\pi}{4}\right), k = 0, 1, 2, 3$

iii $\frac{1}{2}\text{ cis}\left(-\frac{\pi}{10} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$