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Matrices and Systems of Equations - Lesson 4

Matrix Multiplication

LI

- Multiply matrices.
- Take powers of a matrix.
- Orthogonality.
- Matrix Properties 2.

SC

- Primary school arithmetic.

Jan 10-12:11

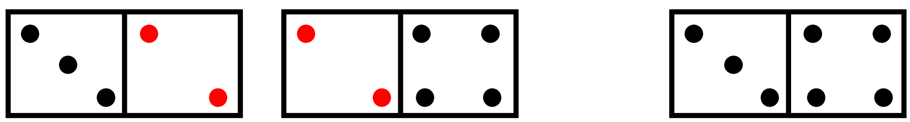
The matrix product $C = AB$ of the $m \times n$ matrix A and the $n \times p$ matrix B is obtained by :

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

C is an $m \times p$ matrix

Note that the matrix product AB only makes sense provided that the number of columns of A equals the number of rows of B.

Think of dominoes :



C_{ij} is obtained by taking the scalar product of the i^{th} row of A with the j^{th} column of B.

In the matrix product AB, A is post-multiplied by B and B is pre-multiplied by A.

MAJOR WARNING :

In general,

$$AB \neq BA$$

Powers of a matrix can now be taken.

The n^{th} power of A is the matrix :

$$A^n = \underbrace{A \times A \times A \ldots \times A}_{n \text{ times}}$$

An $n \times n$ matrix A is orthogonal if :

$$A A^T = A^T A = I_n$$

Sep 24-12:04

Matrix Properties - 2

8) $(AB)C = A(BC)$

9) $A(B + C) = AB + AC$

10) $(A + B)C = AC + BC$

11) $AI = A = IA$

12) $(AB)^T = B^T A^T$

Example 1

Find $\begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$.

$$2 \times 2 \text{ and } 2 \times 3 \rightarrow 2 \times 3$$

$$\begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(1) + (3)(0) & (-1)(4) + (3)(1) & (-1)(-2) + (3)(6) \\ (4)(1) + (6)(0) & (4)(4) + (6)(1) & (4)(-2) + (6)(6) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & -1 & 20 \\ 4 & 22 & 4 \end{pmatrix}}$$

Example 2

Find A^3 if $A = \begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix}$.

$$A^2 = \begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} 1 - 8 & 2 + 0 \\ -4 + 0 & -8 + 0 \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} -7 & 2 \\ -4 & -8 \end{pmatrix}$$

$$A^3 = A \,.\, A^2$$

$$\Rightarrow \quad A^3 = \begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} -7 & 2 \\ -4 & -8 \end{pmatrix}$$

$$\Rightarrow \quad A^3 = \begin{pmatrix} -7 - 8 & 2 - 16 \\ 28 + 0 & -8 + 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{A^3 = \begin{pmatrix} -15 & -14 \\ 28 & -8 \end{pmatrix}}$$

Example 3

If $D = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, show that $D^2 = aD + bI_2$ $(a, b \in \mathbb{R})$.

Hence show that $D^3 = 36D - 91I_2$.

$$D^2 = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$$

$$\Rightarrow \quad D^2 = \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = a\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2a + b & -a \\ 3a & 5a + b \end{pmatrix}$$

$$\Rightarrow \quad \underline{a = 7,\ b = -13}$$

$$\therefore \quad \boxed{D^2 = 7D - 13I_2}$$

$$D^3 = DD^2$$

$$\therefore \quad D^3 = D(7D - 13I_2)$$

$$\Rightarrow \quad D^3 = 7D^2 - 13D$$

$$\Rightarrow \quad D^3 = 7(7D - 13I_2) - 13D$$

$$\Rightarrow \quad D^3 = 49D - 91I_2 - 13D$$

$$\Rightarrow \quad \boxed{D^3 = 36D - 91I_2}$$

Sep 12-14:22

Example 4

Show that $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

$$R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\therefore \quad R^T R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow \quad R^T R = \begin{pmatrix} c^2 + s^2 & -sc + sc \\ -sc + sc & c^2 + s^2 \end{pmatrix} \qquad \boxed{\begin{array}{l} c = \cos\theta \\ s = \sin\theta \end{array}}$$

$$\Rightarrow \quad R^T R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$\boxed{\text{As } R^T R = I_2 \text{, } R \text{ is orthogonal}}$

Example 5

Prove that $A(B + C) = AB + AC$.

Considering the $(i, j)^{th}$ entry of $A(B + C)$, we have,

$$(A(B + C))_{ij} = \sum_{k=1}^{n} A_{ik}(B + C)_{kj}$$

$$= \sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj})$$

$$= \sum_{k=1}^{n} (A_{ik}B_{kj} + A_{ik}C_{kj})$$

$$= \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj}$$

$$= (AB)_{ij} + (AC)_{ij}$$

$$\therefore \quad \boxed{A(B + C) = AB + AC}$$

AH Maths - MiA (2nd Edn.)

- pg. 235-6 Ex. 13.3 Q 1 - 5.
- pg. 238-9 Ex. 13.5 Q 1 - 10.

Ex. 13.3

1 Evaluate

a $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}$ b $\begin{pmatrix}6 & 2\end{pmatrix}\begin{pmatrix}-1\\5\end{pmatrix}$ c $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}3\\2\\1\end{pmatrix}$

d $\begin{pmatrix}2 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}$ e $\begin{pmatrix}2 & 2 & 3 & -1\end{pmatrix}\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}$ f $\begin{pmatrix}6 & 8 & 4 & 2\end{pmatrix}\begin{pmatrix}1\\3\\5\\7\end{pmatrix}$

g $\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ h $\begin{pmatrix}2 & 3 & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

2 Evaluate each matrix product.

a $\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$ b $\begin{pmatrix}3 & 1\\4 & 3\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}$ c $\begin{pmatrix}2 & -1\\4 & 3\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}$

d $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}$ e $\begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix}$ f $\begin{pmatrix}2 & 3\\4 & -1\end{pmatrix}\begin{pmatrix}5\\-7\end{pmatrix}$

g $\begin{pmatrix}2 & 3\\-1 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ h $\begin{pmatrix}2 & -1\\3 & 7\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}$ i $\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & -2\\2 & -1\end{pmatrix}$

j $\begin{pmatrix}2 & 0\\4 & 1\end{pmatrix}\begin{pmatrix}0 & -2\\1 & 2\end{pmatrix}$ k $\begin{pmatrix}2 & 1 & 3\\3 & 1 & 2\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix}$ l $\begin{pmatrix}3 & 1 & 1\\-1 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 2\\0 & 1\end{pmatrix}$

m $\begin{pmatrix}1 & 3 & 2\\-2 & 0 & 1\\4 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}$ n $\begin{pmatrix}1 & 1 & 0\\2 & 0 & 1\\3 & 4 & 5\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}$ o $\begin{pmatrix}1 & 1 & 0\\2 & 0 & 1\\3 & 4 & 5\end{pmatrix}\begin{pmatrix}2 & 1\\-2 & 2\\1 & 0\end{pmatrix}$

p $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}$

3 Simplify these matrix expressions.

a $\begin{pmatrix}\cos\theta & \sin\theta\\\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ b $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta\\\cos\theta\end{pmatrix}$ c $\begin{pmatrix}\cos\theta & \sin\theta\\\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\\sin\theta & -\cos\theta\end{pmatrix}$

4 Given that $A = \begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}$ calculate a A^2 b A^3

5 Express each product as a single matrix.

a $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\-1 & 2\end{pmatrix}$ b $\begin{pmatrix}1 & 2\\4 & -1\end{pmatrix}\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}$ c $\begin{pmatrix}1 & 2\\5 & 7\end{pmatrix}\begin{pmatrix}-1 & 2\\3 & -2\end{pmatrix}$

d $\begin{pmatrix}0 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-1 & 3\end{pmatrix}$ e $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}2 & 7\\1 & 3\end{pmatrix}$ f $\begin{pmatrix}1 & 3\\2 & -1\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}$

Ex. 13.5

1 Simplify

a $\begin{pmatrix}4 & 1 & 5\\2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 3\\3 & 2\\3 & 1\end{pmatrix}$ b $\begin{pmatrix}3 & 1 & 4\\1 & 2 & 5\end{pmatrix}\begin{pmatrix}4 & 1 & 2\\1 & 3 & 1\\7 & 4 & 3\end{pmatrix}$ c $\begin{pmatrix}1 & 0 & -1\\1 & 1 & -1\\-1 & 1 & 0\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\2 & 1 & -1\\0 & 2 & 2\end{pmatrix}$

2 Given $A = \begin{pmatrix}4 & 2 & 1\\3 & 0 & -2\\6 & 5 & 1\end{pmatrix}$, $B = \begin{pmatrix}4 & 7 & 2\\1 & -1 & 0\\2 & 3 & -1\end{pmatrix}$, $C = \begin{pmatrix}2 & 1 & 3\\0 & 1 & -1\\-2 & 3 & 0\end{pmatrix}$ and

$D = \begin{pmatrix}2 & 1 & 0\\3 & 0 & -1\\2 & 4 & 7\end{pmatrix}$ evaluate a AB b CD c $AB + D^2$ d C^2 e C^3 f $AB + CD$

3 a Given that $R = \begin{pmatrix}2 & -1 & 1\\-3 & 4 & -3\\-5 & 5 & -4\end{pmatrix}$ and $S = \begin{pmatrix}-1 & 1 & -1\\3 & -3 & 3\\5 & -5 & 5\end{pmatrix}$, show that $RS = O$.

b Does $SR = O$?

4 Given that $A = \begin{pmatrix}2 & 1\\-1 & 3\end{pmatrix}$ and $B = \begin{pmatrix}3 & 1\\-2 & 4\end{pmatrix}$ find which of these are true.

a $AB = BA$

b $A^2 - B^2 = (A + B)(A - B)$

c $(A + B)^2 = A^2 + 2AB + B^2$

d $(A - B)^2 = A^2 - 2AB + B^2$

5 Explore the claim that for any matrix A, both AA' and $A'A$ are symmetric.

6 a Given that $A = \begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}$ and $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ show that $A^2 = 5I + 2A$.

b Multiplying this equation by A, express A^3 in the form $aI + bA$.
[Hint: you know an expression for A^2.]

7 a If $A = \begin{pmatrix}3 & 4\\2 & 1\end{pmatrix}$ show that $A^2 = 4A + 5I$.

b Hence show that $A^3 = 21A + 20I$.

8 a If $B = \begin{pmatrix}2 & 3\\-1 & 2\end{pmatrix}$ show that $B^2 = 4B - 7I$.

b Express B^3 in the form $aB + bI$ where $a, b \in R$.

9 a Show that $B^2 = \begin{pmatrix}17 & 8\\16 & 9\end{pmatrix}$ where $B = \begin{pmatrix}3 & 2\\4 & 1\end{pmatrix}$.

b By considering the equation $a\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + b\begin{pmatrix}3 & 2\\4 & 1\end{pmatrix} = \begin{pmatrix}17 & 8\\16 & 9\end{pmatrix}$
show that $B^2 = aI + bB$ and find the values of a and b.

c Use this to express B^3, B^4 and B^5 in terms of I and B only.

10 Given $C = \begin{pmatrix}2 & 5\\-3 & 1\end{pmatrix}$ find p and q such that $C^2 = pC + qI$.

Jun 19-11:45

Answers to AH Maths (MiA), pg. 235-6, Ex. 13.3

1 **a** (5) **b** (4) **c** (10)

d (1) **e** (-5) **f** (64)

g $(2x+3y)$ **h** $(2x+3y-4z)$

2 **a** $\begin{pmatrix} 4 \\ 11 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$

d $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 8 \\ -5 \end{pmatrix}$ **f** $\begin{pmatrix} -11 \\ 27 \end{pmatrix}$

g $\begin{pmatrix} 2x+3y \\ 5y-x \end{pmatrix}$ **h** $\begin{pmatrix} 2p-q \\ 3p+7q \end{pmatrix}$ **i** $\begin{pmatrix} 6 & -4 \\ 0 & 1 \end{pmatrix}$

j $\begin{pmatrix} 0 & -4 \\ 1 & -6 \end{pmatrix}$ **k** $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ **l** $\begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}$

m $\begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$ **n** $\begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$ **o** $\begin{pmatrix} 0 & 3 \\ 5 & 2 \\ 3 & 11 \end{pmatrix}$

p $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

3 **a** $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4 **a** $\begin{pmatrix} -1 & 4 \\ -2 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} -5 & 2 \\ -1 & -5 \end{pmatrix}$

5 **a** $\begin{pmatrix} 1 & 5 \\ 5 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 5 \\ 7 & -7 \end{pmatrix}$ **c** $\begin{pmatrix} 5 & -2 \\ 16 & -4 \end{pmatrix}$

d $\begin{pmatrix} -1 & 3 \\ 1 & 7 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ **f** $\begin{pmatrix} a+3c & b+3d \\ 2a-c & 2b-d \end{pmatrix}$

Answers to AH Maths (MiA), pg. 238-9, Ex. 13.5

1 **a** $\begin{pmatrix} 22 & 19 \\ 14 & 11 \end{pmatrix}$ **b** $\begin{pmatrix} 41 & 22 & 19 \\ 41 & 27 & 19 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -3 & -3 \\ 4 & -2 & -4 \\ 0 & 2 & 0 \end{pmatrix}$

2 **a** $\begin{pmatrix} 20 & 29 & 7 \\ 8 & 15 & 8 \\ 31 & 40 & 11 \end{pmatrix}$ **b** $\begin{pmatrix} 13 & 14 & 20 \\ 1 & -4 & -8 \\ 5 & -2 & -3 \end{pmatrix}$ **c** $\begin{pmatrix} 27 & 31 & 6 \\ 12 & 14 & 1 \\ 61 & 70 & 56 \end{pmatrix}$

d $\begin{pmatrix} -2 & 12 & 5 \\ 2 & -2 & -1 \\ -4 & 1 & -9 \end{pmatrix}$ **e** $\begin{pmatrix} -14 & 25 & -18 \\ 6 & -3 & 8 \\ 10 & -30 & -13 \end{pmatrix}$

f $\begin{pmatrix} 33 & 43 & 27 \\ 9 & 11 & 0 \\ 36 & 38 & 8 \end{pmatrix}$

3 **a** Proof **b** Yes

4 All false

5 True; proof

6 **a** Proof **b** $A^3 = 10I + 9A$

7 **a** Proof **b** Proof

8 **a** Proof **b** $B^3 = 9B - 28I$

9 **a** Proof **b** $B^2 = 4B + 5I$

c $B^3 = 21B + 20I$; $B^4 = 104B + 105I$; $B^5 = 521B + 520I$

10 $p = 3$; $q = -17$