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Unit 2 : Proof by Mathematical Induction - Lesson 4

Mathematical Induction 4 (Differentiation)

LI

- Use Proof by Mathematical Induction to solve problems involving differentiation.

SC

- Product Rule.

Proof by Mathematical Induction

The Principle of Mathematical Induction (PMI) states that, to prove a statement ($P(n)$) about an infinite set of natural numbers :

- Prove the **Base Case** : $P(n_0)$ is true.
- Prove the **Inductive Step** :
 $P(k)$ true $\Rightarrow P(k + 1)$ true.

Then $P(n)$ is true $\forall n \geq n_0$.

Usually, $n_0 = 1$; then we would state the conclusion as ' $P(n)$ is true $\forall n \in \mathbb{N}$ '.

Notation for Higher Derivatives

For a function $y = f(x)$, the n^{th} derivative of f with respect to x is variously denoted as :

$$\frac{d^n}{dx^n} f$$

$$D^n f$$

The $(k + 1)^{\text{th}}$ derivative of f can be written in terms of the k^{th} derivative of f :

$$\frac{d^{k+1}}{dx^{k+1}} f = \frac{d}{dx} \left(\frac{d^k}{dx^k} f \right) ; \quad \frac{d}{dx} \text{ means } \frac{d^1}{dx^1}$$

$$D^{k+1} f = D(D^k f) ; \quad D \text{ means } D^1$$

Example 1

Prove by induction that $\frac{d^n}{dx^n} e^{5x} = 5^n e^{5x} \quad \forall n \in \mathbb{N}$.

$$P(n) : \frac{d^n}{dx^n} e^{5x} = 5^n e^{5x}$$

Base Case

$$LHS = \frac{d^1}{dx^1} e^{5x} = 5e^{5x}$$

$$RHS = 5^1 e^{5x} = 5e^{5x}$$

As LHS = RHS, P(1) is true

Inductive Step

Assume P(k) is true for some $k \in \mathbb{N}$, i.e. assume that :

$$D^k(e^{5x}) = 5^k e^{5x} \quad \xrightarrow{\text{Inductive Hypothesis}}$$

RTP statement

$$D^{k+1}(e^{5x}) = 5^{k+1} e^{5x}$$

$$D^{k+1}(e^{5x}) = D(D^k(e^{5x}))$$

$$\Rightarrow D^{k+1}(e^{5x}) = D(5^k e^{5x})$$

$$\Rightarrow D^{k+1}(e^{5x}) = 5^k D(e^{5x})$$

$$\Rightarrow D^{k+1}(e^{5x}) = 5^k \cdot 5 \cdot e^{5x}$$

$$\Rightarrow D^{k+1}(e^{5x}) = 5^{k+1} e^{5x}$$

Hence, P(k) true \Rightarrow P(k + 1) true

'P(1) true' and 'P(k) true \Rightarrow P(k + 1) true' together imply, by the PMI, that P(n) is true $\forall n \in \mathbb{N}$

Example 2

Prove by induction that $\frac{d^n}{dx^n} (x e^x) = (n + x) e^x \quad \forall n \in \mathbb{N}$.

$$P(n): D^n(x e^x) = (n + x) e^x$$

Base Case

$$LHS = D^1(x e^x) = 1 \cdot e^x + x \cdot e^x = (1 + x) e^x$$

$$RHS = (1 + x) e^x$$

As LHS = RHS, $P(1)$ is true

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$D^k(x e^x) = (k + x) e^x \quad \leftarrow \begin{matrix} \text{Inductive} \\ \text{Hypothesis} \end{matrix}$$

RTP statement

$$D^{k+1}(x e^x) = (k + 1 + x) e^x$$

$$D^{k+1}(x e^x) = D(D^k(x e^x))$$

$$\Rightarrow D^{k+1}(x e^x) = D((k + x) e^x)$$

$$\Rightarrow D^{k+1}(x e^x) = 1 \cdot e^x + (k + x) \cdot e^x$$

$$\Rightarrow D^{k+1}(x e^x) = e^x (1 + k + x)$$

$$\Rightarrow D^{k+1}(x e^x) = (k + 1 + x) e^x$$

Hence, $P(k)$ true $\Rightarrow P(k + 1)$ true

' $P(1)$ true' and ' $P(k)$ true $\Rightarrow P(k + 1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

Example 3

Prove by induction that $\frac{d^n}{dx^n} \ln x = (-1)^{n-1} (n-1)! x^{-n}$ for all natural numbers.

$$P(n) : D^n (\ln x) = (-1)^{n-1} (n-1)! x^{-n}$$

Base Case

$$LHS = D^1 (\ln x) = 1/x = x^{-1}$$

$$RHS = (-1)^{1-1} (1-1)! x^{-1} = 1 \cdot 0! \cdot x^{-1} = x^{-1}$$

As LHS = RHS, P(1) is true

Inductive Step

Assume P(k) is true for some $k \in \mathbb{N}$, i.e. assume that :

$$D^k (\ln x) = (-1)^{k-1} (k-1)! x^{-k} \quad \leftarrow \text{Inductive Hypothesis}$$

RTP statement

$$D^{k+1} (\ln x) = (-1)^k k! x^{-(k+1)}$$

$$\begin{aligned} & D^{k+1} (\ln x) = D(D^k (\ln x)) \\ \Rightarrow & D^{k+1} (\ln x) = D((-1)^{k-1} (k-1)! x^{-k}) \\ \Rightarrow & D^{k+1} (\ln x) = (-1)^{k-1} (k-1)! D(x^{-k}) \\ \Rightarrow & D^{k+1} (\ln x) = (-1)^{k-1} (k-1)! (-k x^{-k-1}) \\ \Rightarrow & D^{k+1} (\ln x) = (-1)^{k-1} (k-1)! (-1) \cdot k \cdot x^{-k-1} \\ \Rightarrow & D^{k+1} (\ln x) = (-1)^{k-1+1} k \cdot (k-1)! x^{-(k+1)} \\ \Rightarrow & D^{k+1} (\ln x) = (-1)^k k! x^{-(k+1)} \end{aligned}$$

Hence, P(k) true \Rightarrow P(k + 1) true

'P(1) true' and 'P(k) true \Rightarrow P(k + 1) true' together imply, by the PMI, that P(n) is true $\forall n \in \mathbb{N}$

Example 4

Prove by induction that $D^n ((1 - x)^{-1}) = n! (1 - x)^{-(n+1)}$

for all natural numbers.

$$P(n) : D^n ((1 - x)^{-1}) = n! (1 - x)^{-(n+1)}$$

Base Case

$$LHS = D^1 ((1 - x)^{-1}) = (-1)(1 - x)^{-2}(-1) = (1 - x)^{-2}$$

$$RHS = 1! (1 - x)^{-(1+1)} = (1 - x)^{-2}$$

As LHS = RHS, $P(1)$ is true

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$D^k ((1 - x)^{-1}) = k! (1 - x)^{-(k+1)} \quad \leftarrow \begin{matrix} \text{Inductive} \\ \text{Hypothesis} \end{matrix}$$

RTP statement

$$D^{k+1} ((1 - x)^{-1}) = (k + 1)! (1 - x)^{-(k+2)}$$

$$D^{k+1} ((1 - x)^{-1}) = D(D^k ((1 - x)^{-1}))$$

$$\Rightarrow D^{k+1} ((1 - x)^{-1}) = D(k! (1 - x)^{-(k+1)})$$

$$\Rightarrow D^{k+1} ((1 - x)^{-1}) = k! D((1 - x)^{-k-1})$$

$$\Rightarrow D^{k+1} ((1 - x)^{-1}) = k! (-k - 1) \cdot (1 - x)^{-k-1-1} \cdot (-1)$$

$$\Rightarrow D^{k+1} ((1 - x)^{-1}) = k! (-k + 1) \cdot (1 - x)^{-k-2} \cdot (-1)$$

$$\Rightarrow D^{k+1} ((1 - x)^{-1}) = (k + 1)! (1 - x)^{-k-2}$$

Hence, $P(k)$ true $\Rightarrow P(k + 1)$ true

' $P(1)$ true' and ' $P(k)$ true $\Rightarrow P(k + 1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

Example 5

Prove by induction that $D^n(\sin x) = \sin(x + (n\pi/2)) \quad \forall n \in \mathbb{N}$.

$$P(n): D^n(\sin x) = \sin(x + (n\pi/2))$$

Base Case

$$LHS = D^1(\sin x) = \cos x$$

$$\begin{aligned} RHS &= \sin(x + (\pi/2)) = \sin x \cos(\pi/2) + \sin(\pi/2) \cos x \\ &= \sin x \cdot 0 + 1 \cdot \cos x \\ &= \cos x \end{aligned}$$

As LHS = RHS, P(1) is true

Inductive Step

Assume P(k) is true for some $k \in \mathbb{N}$, i.e. assume that:

$$D^k(\sin x) = \sin(x + (k\pi/2)) \leftarrow \text{Inductive Hypothesis}$$

RTP statement

$$D^{k+1}(\sin x) = \sin(x + ((k+1)\pi/2))$$

$$D^{k+1}(\sin x) = D(D^k(\sin x))$$

$$\Rightarrow D^{k+1}(\sin x) = D(\sin(x + (k\pi/2)))$$

$$\Rightarrow D^{k+1}(\sin x) = \cos(x + (k\pi/2)) \cdot 1$$

$$\Rightarrow D^{k+1}(\sin x) = \sin(x + (k\pi/2) + \pi/2)$$

$$\Rightarrow D^{k+1}(\sin x) = \sin(x + ((k+1)\pi/2))$$

Hence, P(k) true \Rightarrow P(k + 1) true

'P(1) true' and 'P(k) true \Rightarrow P(k + 1) true' together imply, by the PMI, that P(n) is true $\forall n \in \mathbb{N}$

Questions (and 'Answers' !)

Prove by mathematical induction that $\forall n \in \mathbb{N}$:

$$1) D^n(e^{7x}) = 7^n e^{7x} \quad (\forall n \in \mathbb{N}).$$

$$2) D^n(x e^{-x}) = (-1)^n (x - n) e^{-x} \quad (\forall n \in \mathbb{N}).$$

$$3) D^n((1 - 3x)^{-1}) = 3^n n! (1 - 3x)^{-(n+1)} \quad (\forall n \in \mathbb{N}).$$

$$4) D^n(x \ln x) = (-1)^n (n - 2)! x^{-(n-1)} \quad (\forall n \geq 2).$$

$$5) D^n(\cos x) = \cos(x + (n\pi/2)) \quad (\forall n \in \mathbb{N}).$$