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Vectors, Lines and Planes - Lesson 4

Intersections of Lines and Planes

LI

- Find intersections of lines and planes.
- Find intersections of 2 and 3 planes.
- Find the angle between 2 planes and the angle between a line and a plane.

SC

- Scalar product.
- Equations of lines and planes.
- Gaussian elimination.

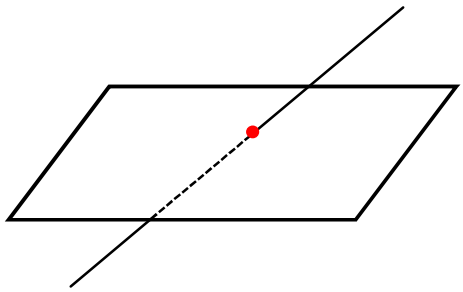
Jan 10-12:11

Intersection of a Line and a Plane

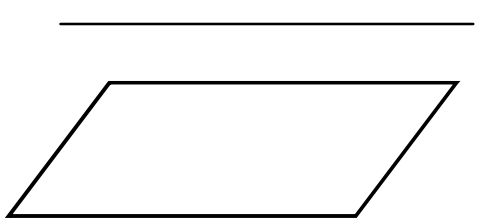
Get line in parametric form and plane in Cartesian form; substitute parametric equations for x, y and z into plane equation. An equation of the following form will result :

$$At + B = 0$$

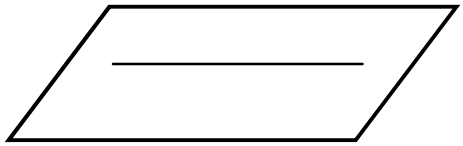
1 Intersection Point ($A \neq 0$) :



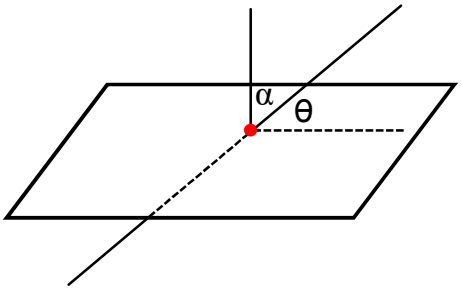
No Intersection ($A = 0, B \neq 0$) :



Infinitely Many Intersection Points ($A = 0, B = 0$) :



The angle between a line and a plane is the complement of the angle between the line direction and plane normal :



Angle between plane normal and line direction is α

Angle between line and plane is θ

Mar 11-13:04

Intersection of 2 Planes

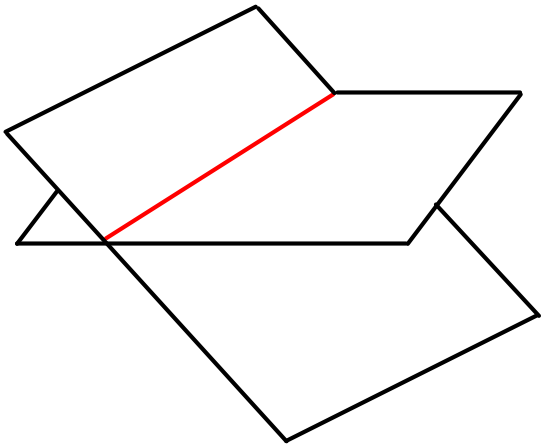
This involves solving a pair of equations for x, y and z; the augmented matrix, after row-reduction, can take various forms.

Intersection in a Plane :

$$\left(\begin{array}{ccc|c} a & b & c & p \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At least one of $a, b, c \neq 0$

Intersection in a Line :

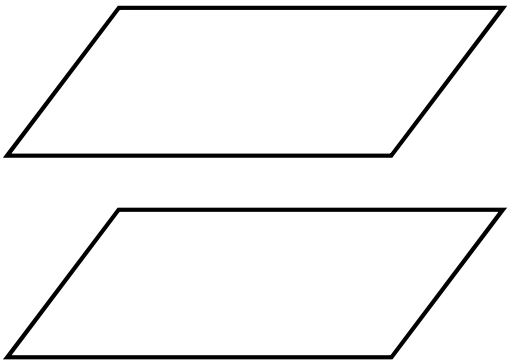


$$\left(\begin{array}{ccc|c} a & b & c & p \\ 0 & e & f & q \end{array}\right)$$

At least one of $a, b, c \neq 0$

At least one of $e, f \neq 0$

No Intersection :



$$\left(\begin{array}{ccc|c} a & b & c & p \\ 0 & 0 & 0 & q \end{array}\right)$$

At least one of $a, b, c \neq 0$

$q \neq 0$

The angle between 2 planes is the angle between their normal vectors

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Intersection(s) of 3 Planes

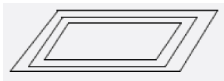
This involves solving 3 equations for x, y and z; the augmented matrix, after row-reduction, can take various forms.

Type 1 :

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At least one of $a, b, c \neq 0$

3 coincident planes



Solutions : infinitely many points in a plane

Type 2 :

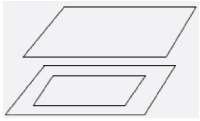
$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At least one of $a, b, c \neq 0$

$k \neq 0$

(a) 2 plane equations the same

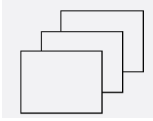
2 coincident planes, 1 different from these



Solutions : none

(b) No plane equations the same

3 distinct planes



Solutions : none

Type 3 :

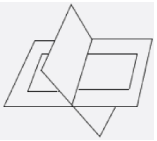
$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At least one of $a, b, c \neq 0$

At least one of $e, f \neq 0$

(a) 2 distinct normals

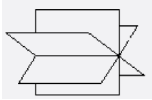
2 coincident planes, 1 different from these



Solutions : 1 line

(b) 3 distinct normals

3 distinct planes



Solutions : 1 line

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Type 4 :

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right)$$

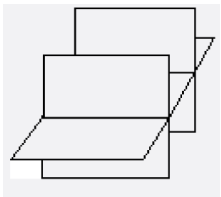
At least one of $a, b, c \neq 0$

At least one of $e, f \neq 0$

$l \neq 0$

(a) 2 distinct normals

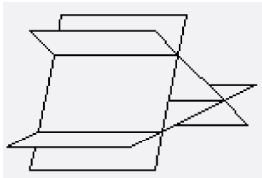
2 parallel planes,
1 intersecting both these



Solutions : 2 lines

(b) 3 distinct normals

3 distinct planes



Solutions : 3 lines

Type 5 :

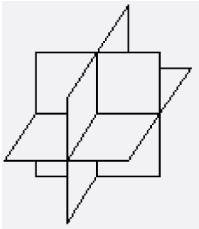
$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right)$$

At least one of $a, b, c \neq 0$

At least one of $e, f \neq 0$

$i \neq 0$

3 distinct planes



Solutions : 1 point

Example 1

Find the point of intersection of the line,

$$\frac{x - 7}{3} = \frac{y - 11}{4} = \frac{z - 24}{13}$$

with the plane $6x + 4y - 5z = 28$. Also determine the angle between the line and the plane.

Substituting the parametric line equations,

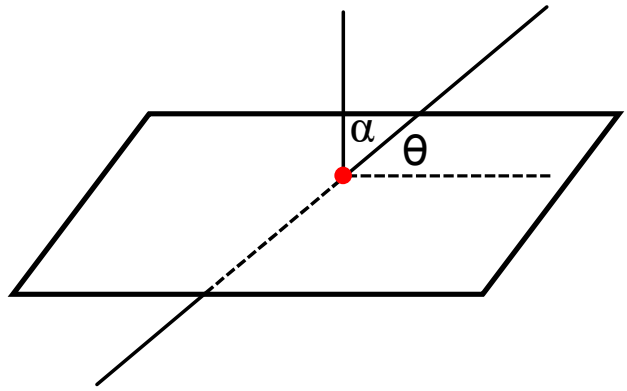
$$x = 3t + 7$$
$$y = 4t + 11$$
$$z = 13t + 24$$

into the plane equation gives, upon simplification (check !),

$$-31t = 62$$
$$\Rightarrow \quad \underline{t = -2}$$

Substituting this value of t into the parametric line equations gives the intersection point of the line and the plane.

$$\boxed{\text{Intersection point : } (1, 3, -2)}$$



Using the line direction vector $(3, 4, 13)^T$ and the plane normal vector $(6, 4, -5)^T$ gives,

$$\cos\alpha = -\frac{31}{\sqrt{194}\,\sqrt{77}}$$

$$\Rightarrow \quad \underline{\alpha = 75.3^\circ}$$

$$\theta = 90^\circ - 75.3^\circ$$

$$\Rightarrow \quad \boxed{\theta = 14.7^\circ}$$

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Example 2

Show that the line,

$$\frac{x - 3}{12} = \frac{y + 4}{-4} = \frac{z - 1}{3}$$

and the plane $2x + 3y - 4z = -10$ intersect in a line.

Substituting the parametric line equations,

$$x = 12t + 3$$
$$y = -4t - 4$$
$$z = 3t + 1$$

into the plane equation gives, upon simplification (check !),

$$\underline{0 = 0}$$

$\therefore$ $\boxed{\text{Infinitely many intersection points}}$

Feb 15-07:43

Example 3

Find the intersection of the 3 planes,

$$\begin{aligned} x - y + z &= 10 \\ 2x - y + 3z &= 5 \\ 4x - 2y + 6z &= 10 \end{aligned}$$

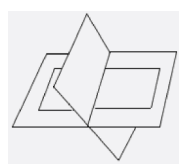
Notice that the last 2 equations are identical. The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 2 & -1 & 3 & 5 \\ 4 & -2 & 6 & 10 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 2 & -1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 1 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The form of the matrix, and the fact that there are 2 distinct normals, shows that we are in Type 3 (a).



Let $z = t$. Then the row-reduced equations become,

$$\begin{aligned} x - y + t &= 10 \\ y + t &= -15 \Rightarrow \underline{y = -15 - t} \\ \therefore \quad x &= 10 - t - 15 - t \\ \Rightarrow \quad \underline{x} &\underline{= -5 - 2t} \end{aligned}$$

$$\boxed{\text{Intersection : the line } \frac{x+5}{-2} = \frac{y+15}{-1} = \frac{z}{1} \ (= t \in \mathbb{R})}$$

AH Maths - MiA (2nd Edn.)

- pg. 300-1 Ex. 15.10
  Q 1 a, b, 2 c, 4 a.
- pg. 303 Ex. 15.12 Q 1.
- pg. 307 Ex. 15.13
  Q 1 a, b, c, 2 a, b.

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Ex. 15.10

1 For each of these plane and line pairs find

i the coordinates of the point of intersection

ii the angle between the line and the plane.

a $\frac{x-12}{5} = \frac{y+7}{-4} = \frac{z-5}{3}$; $5x + 3y - z = 14$

b $x - 3 = 2t, y + 3 = 0, z - 2 = -t$; $3x - 4y + 2z = 33$

2 Show that in each case the given line is parallel to the given plane.

c $\frac{x-9}{3} = \frac{y-4}{2} = \frac{z+3}{-1}$ and $3x - 4y + z = 9$

4 Show that in each case the given line lies within the given plane.

a $\frac{x-3}{12} = \frac{y+4}{-4} = \frac{z-1}{3}$ and $2x + 3y - 4z + 10 = 0$

Ex. 15.12

1 Find the equations of the line of intersection of each of these pairs of planes.

a $x - y - 3z + 7 = 0$ and $2x + 3y - z + 4 = 0$

b $x - 2y - 2z + 8 = 0$ and $2x - y - 2z = 1$

Ex. 15.13

1 Determine how each of these sets of three planes intersect. Give the coordinates of the point of intersection or the equations of any lines of intersection.

a
$$\begin{aligned} x + 2y + 3z &= 3 \\ 2x - y + 4z &= 5 \\ x - 3y + 2z &= 2 \end{aligned}$$

b
$$\begin{aligned} x + 2y + 3z &= 3 \\ 2x - y + 4z &= 5 \\ x - 3y + z &= 2 \end{aligned}$$

c
$$\begin{aligned} 2x + y + 3z &= 5 \\ 3x - y + 2z &= 1 \\ x - 2y - z &= -3 \end{aligned}$$

2 Comment on the intersection of these sets of planes.

a
$$\begin{aligned} x + 4y + 5z &= 6 \\ 2x + 8y + 10z &= 12 \\ 8x + 32y + 40z &= 48 \end{aligned}$$

b
$$\begin{aligned} 2x - 4y + 4z &= 8 \\ x - 2y + 2z &= 7 \\ 3x - 6y + 6z &= -2 \end{aligned}$$

Answers to AH Maths (MiA), pg. 300-1, Ex. 15.10

1 a $(2, 1, -1)$ $13{\cdot}8°$ b $(7, -3, 0)$ $19{\cdot}4°$

2 Proof, show that the direction vector of the line is perpendicular to the normal vector to the plane.

4 Proof (as Q2 but also show that "the" point on the line lies in the plane)

Answers to AH Maths (MiA), pg. 303, Ex. 15.12

1 a $\frac{x+5}{2} = \frac{y-2}{-1} = \frac{z}{1}$ b $\frac{x}{2} = \frac{y-9}{-2} = \frac{z+5}{3}$

Answers to AH Maths (MiA), pg. 307, Ex. 15.13

1 a In the single point $\left(\frac{13}{5}, \frac{1}{5}, 0\right)$

b In the line $\frac{13-5x}{11} = \frac{1-5y}{2} = \frac{z}{1}$

c In three parallel lines

$$\frac{5x+4}{-5} = \frac{5y+17}{-5} = \frac{z}{1}$$

$$\frac{5x+13}{-5} = \frac{5y-1}{-5} = \frac{z}{1}$$

$$\frac{5x-5}{-5} = \frac{5y-10}{-5} = \frac{z}{1}$$

2 a Three coincident planes

b Three parallel planes