

Integration - Lesson 4

Integrals of Sine and Cosine

LI

- Know the integrals of $\sin(ax + b)$ and $\cos(ax + b)$.

SC

- Opposite of Chain Rule.

Important Reminder

When differentiating sine and cosine, we work in radians.

Applying the Chain Rule for differentiation to the functions,

$$y_1 = \frac{1}{a} \sin(ax + b) \quad \text{and} \quad y_2 = -\frac{1}{a} \cos(ax + b)$$

(a and b are constants with $a \neq 0$)

we get,

$$\frac{dy_1}{dx} = \cos(ax + b) \quad \text{and} \quad \frac{dy_2}{dx} = \sin(ax + b)$$

Remember, these derivatives are only true when x is in RADIANS

Hence,

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$$

and

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$$

($a \neq 0$)

(x is in RADIANS for both)

Common special cases of these occur when $a = 1$ and $b = 0$:

$$\int \sin x dx = -\cos x + C$$

and

$$\int \cos x dx = \sin x + C$$

(x is in RADIANS for both)

Example 1

Integrate $3 \cos x$.

$$\begin{aligned}\int 3 \cos x \, dx &= 3 \int \cos x \, dx \\ &= \boxed{3 \sin x + C}\end{aligned}$$

Example 2

Find $-\frac{3}{5} \int \sin x \, dx.$

$$\begin{aligned}-\frac{3}{5} \int \sin x \, dx &= -\frac{3}{5} (-\cos x) + C \\&= \boxed{\frac{3}{5} \cos x + C}\end{aligned}$$

Example 3

Integrate $8 \cos 16x$.

$$\begin{aligned}\int 8 \cos 16x \, dx &= 8 \int \cos 16x \, dx \\ &= \frac{8}{16} \sin 16x + C \\ &= \boxed{\frac{1}{2} \sin 16x + C}\end{aligned}$$

Example 4

Find $\frac{4}{5} \int \sin(5x - 1) dx.$

$$\begin{aligned}& \frac{4}{5} \int \sin(5x - 1) dx \\&= \frac{4}{5} \left(-\frac{1}{5} \right) \cos(5x - 1) + C\end{aligned}$$

$$= \boxed{-\frac{4}{25} \cos(5x - 1) + C}$$

CfE Higher Maths

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Q 1 a-o, 2 a-h, 3 - 5

Questions

1 Integrate with respect to x .

a $8\cos x$

b $3\sin x$

c $-4\sin x$

d $2\cos x$

e $\frac{3}{2}\cos x$

f $-\frac{5\sin x}{4}$

g $4\cos\left(x - \frac{\pi}{3}\right)$

h $5\sin(x - 2)$

i $\cos 5x$

j $\sin 4x$

k $8\cos 2x$

l $\frac{1}{2}\cos(3x - \pi)$

m $\sin\frac{1}{2}x$

n $6\cos\frac{3}{2}x$

o $9\sin 5x$

2 Integrate with respect to x .

a $3\sin x + 5x^2$

b $2\cos x + \frac{3}{x^2}$

c $4\sqrt{x} - \sin 2x$

d $\sin\left(x - \frac{\pi}{6}\right) + (x - 3)^5$

e $(4x + 1)^5 - \cos 3x$

f $\frac{x - 1}{x^3} + 4\sin(x - 1)$

g $\frac{2}{\sqrt{x}} - 5\cos 3x$

h $\frac{5}{4x^3} - 4\sin\frac{1}{2}x$

3 a By using the fact that $\cos 2x = 2\cos^2 x - 1$, show that $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

b Using your answer from part **a** show that $\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$.

4 a By using the fact that $\cos 2x = 1 - 2\sin^2 x$, show that $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

b Hence show that $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$.

5 Integrate with respect to x .

a $4\sin^2 x$

b $\frac{1}{2}\cos^2 x$

c $\sin 2x + \sin^2 x$

d $\cos^2 x - \frac{2}{3}x^2$

e $\sin^2 x - \cos^2 x$

f $\frac{2}{5}\cos^2 x$

Answers

- | | |
|---|--|
| 1 a $8\sin x + c$
b $-3\cos x + c$
c $4\cos x + c$
d $2\sin x + c$
e $\frac{3}{2}\sin x + c$
f $\frac{5}{4}\cos x + c$
g $4\sin\left(x - \frac{\pi}{3}\right)$
h $-5\cos(x - 2) + c$
i $\frac{1}{5}\sin 5x + c$
j $-\frac{1}{4}\cos 4x + c$
k $4\sin 2x + c$
l $\frac{1}{6}\sin(3x - \pi) + c$
m $-2\cos\frac{1}{2}x + c$
n $4\sin\left(\frac{3x}{2}\right) + c$
o $-\frac{9}{5}\cos 5x + c$ | 2 a $\frac{5x^3}{3} - 3\cos x + c$
b $-\frac{3}{x} + 2\sin x + c$
c $\frac{8x^{\frac{3}{2}}}{3} + \frac{1}{2}\cos 2x + c$
d $\frac{(x-3)^6}{6} - \cos\left(x - \frac{\pi}{6}\right) + c$
e $\frac{1}{24}(4x+1)^6 - \frac{1}{3}\sin 3x + c$
f $\frac{1-2x}{2x^2} - 4\cos(x-1) + c$
g $4\sqrt{x} - \frac{5}{3}\sin 3x + c$
h $-\frac{5}{8x^2} + 8\cos\frac{1}{2}x + c$ |
|---|--|

3 a $1 + \cos 2x = 2(\cos x)^2$
 $(\cos x)^2 = \frac{1}{2}(1 + \cos 2x)$
 $= \frac{1}{2} + \frac{1}{2}\cos 2x$

b $\int \left(\frac{1}{2} + \frac{1}{2}\cos 2x \right) dx$
 $= \frac{1}{2}x + \frac{1}{2} \int \cos 2x dx$
 $= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

4 a $1 - 2(\sin x)^2 = \cos 2x$
 $-2(\sin x)^2 = (\cos 2x - 1)$
 $(\sin x)^2 = \frac{1}{2} - \frac{1}{2}\cos 2x$

b $\int \left(\frac{1}{2} - \frac{1}{2}\cos 2x \right) dx$
 $= \frac{1}{2}x - \frac{1}{2} \int \cos 2x dx$
 $= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

5 Please note that there are many different forms of the answer, each one correct.

- a $2x - \sin 2x + c$
- b $\frac{1}{4}(x + \sin x \cos x) + c$
- c $\frac{1}{4}(2x - 2\cos 2x - \sin 2x) + c$
- d $\frac{x}{2} + \frac{1}{4}\sin 2x - \frac{2x^3}{9} + c$
- e $-\frac{1}{2}\sin 2x + c$
- f $\frac{1}{5}x + \frac{1}{10}\sin 2x + c$