Graph Sketching

LI
- Sketch graphs of polynomials, especially cubics and quartics.

SC
- SPs.
- Intersections with axes.
- Behaviour as $x \rightarrow \pm \infty$. 
To sketch the graph of a polynomial (indicating stationary points), find:

- **x-intercepts** (put \( y = 0 \) then synthetic division to get roots).
- **y-intercept** (put \( x = 0 \)).
- **SPs and their nature.**
- **Behaviour as** \( x \to \pm \infty \).
Example 1

Sketch the graph of the function
\[ f(x) = x^3 + 2x^2 + x, \] annotating it fully.

'Annotating it fully' means indicating intersections with axes and indicating SPs.

\[ f(x) = x^3 + 2x^2 + x \]
\[ \Rightarrow f(x) = x(x^2 + 2x + 1) \]
\[ \Rightarrow f(x) = x(x + 1)^2 \]

- **x-intercepts:**
  \[ f(x) = 0 \]
  \[ \Rightarrow x(x + 1)^2 = 0 \]
  \[ \Rightarrow x = 0, x = -1 \]
  \[ \therefore (0, 0), (0, -1) \]

- **y-intercept:**
  \[ f(x) = x^3 + 2x^2 + x \]
  \[ \therefore f(0) = (0)^3 + 2(0)^2 + 0 \]
  \[ \Rightarrow f(0) = 0 \]
  \[ \therefore (0, 0) \]

- **SPs:**
  \[ f'(x) = 0 \]
  \[ \therefore 3x^2 + 4x + 1 = 0 \]
  \[ \Rightarrow (3x + 1)(x + 1) = 0 \]
  \[ \Rightarrow x = -\frac{1}{3}, x = -1 \]
\[ x = -1: \]
\[ f(x) = x^3 + 2x^2 + x \]
\[ \therefore f(-1) = (-1)^3 + 2(-1)^2 + (-1) \]
\[ \Rightarrow f(-1) = -1 + 2 - 1 \]
\[ \Rightarrow f(-1) = 0 \]
\[ \therefore (-1, 0) \]

\[ x = -\frac{1}{3}: \]
\[ f(x) = x^3 + 2x^2 + x \]
\[ \therefore f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 2\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right) \]
\[ \Rightarrow f\left(-\frac{1}{3}\right) = -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} \]
\[ \Rightarrow f\left(-\frac{1}{3}\right) = -\frac{4}{27} \]
\[ \therefore \left(-\frac{1}{3}, -\frac{4}{27}\right) \]
\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -2 & -1 & -\frac{1}{2} & -\frac{1}{3} & 0 \\
\hline
f'(x) & + & 0 & - & 0 & + \\
\hline
\text{Slope} & \diagup & \hspace{1cm} & \diagdown & \hspace{1cm} & \diagup \\
\hline
\end{array}
\]

\[
f'(x) = (3x + 1)(x + 1)
\]

\[
\therefore f'(-2) = (-5)(-1)
\]

\[
\Rightarrow f'(-2) = 5 > 0
\]

\[
f'(x) = (3x + 1)(x + 1)
\]

\[
\therefore f'\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)
\]

\[
\Rightarrow f'\left(-\frac{1}{2}\right) = -\frac{1}{4} < 0
\]

\[
f'(x) = (3x + 1)(x + 1)
\]

\[
\therefore f'(0) = (1)(1)
\]

\[
\Rightarrow f'(0) = 1 > 0
\]

\[
(-1, 0) \text{ is a local max. and } \left(-\frac{1}{3}, -\frac{4}{27}\right) \text{ is a local min.}
\]
• Behaviour as $x \to \pm \infty$:

For large values of $x$ (positive or negative), the $x^3$ term is dominant, so that,

$$y = f(x) \approx x^3$$

As $x \to +\infty$ (i.e. for large and positive $x$),

$$f(x) \to +\infty$$ (i.e. $y$ is large and positive).

Positive $x$ (positive)$^3$ = positive

As $x \to -\infty$ (i.e. for large and negative $x$),

$$f(x) \to -\infty$$ (i.e. $y$ is large and negative).

Positive $x$ (negative)$^3$ = negative

$$y = x^3 + 2x^2 + x$$

$(-\frac{1}{3}, -\frac{4}{27})$
Example 2

Example 3
Example 2

Sketch the graph of the function

\[ f(x) = -x^3 - 3x^2 + 9x + 2. \]

By inspection:

\[
\begin{array}{cccc}
  x^3 & x^2 & x^1 & x^0 \\
 2 & -1 & -3 & 9 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
  & -2 & -10 & -2 \\
\end{array}
\]

\[
\begin{array}{cccc}
  -1 & -5 & -1 & 0 \\
\end{array}
\]

⇒ \[ f(x) = (x - 2)(-x^2 - 5x - 1) \]

⇒ \[ f(x) = -(x - 2)(x^2 + 5x + 1) \]

**x-intercepts:**

\[ f(x) = 0 \]

⇒ \( -(x - 2)(x^2 + 5x + 1) = 0 \)

⇒ \[ x = 2, x = \frac{-5 \pm \sqrt{21}}{2} \]

\[ \therefore (2, 0), \left(\frac{-5 \pm \sqrt{21}}{2}, 0\right) \]

**y-intercept:**

\[ f(x) = -x^3 - 3x^2 + 9x + 2 \]

\[ \therefore f(0) = -(0)^3 - 3(0)^2 + 9(0) + 2 \]

⇒ \[ f(0) = 2 \]

\[ \therefore (0, 2) \]

**SPs:**

\[ f'(x) = 0 \]

\[ -3x^2 - 6x + 9 = 0 \]

⇒ \[ -3(x^2 + 2x - 3) = 0 \]

⇒ \[ -3(x + 3)(x - 1) = 0 \]

⇒ \[ x = -3, x = 1 \]
\[ x = -3: \]
\[ f(x) = -x^3 - 3x^2 + 9x + 2 \]
\[ \therefore f(-3) = -(-3)^3 - 3(-3)^2 + 9(-3) + 2 \]
\[ \Rightarrow f(-3) = 27 - 27 - 27 + 2 \]
\[ \Rightarrow f(-3) = -25 \]
\[ : (\ -3, \ -25) \]

\[ x = 1: \]
\[ f(x) = -x^3 - 3x^2 + 9x + 2 \]
\[ \therefore f(1) = -(1)^3 - 3(1)^2 + 9(1) + 2 \]
\[ \Rightarrow f(1) = -1 - 3 + 9 + 2 \]
\[ \Rightarrow f(1) = 7 \]
\[ : (1, 7) \]
\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -4 & -3 & 0 & 1 & 2 \\
\hline
f'(x) & - & 0 & + & 0 & - \\
\hline
\text{Slope} & \diagdown & \text{---} & \diagup & \text{---} & \diagdown \\
\hline
\end{array}
\]

\[
f'(x) = -3(x + 3)(x - 1)
\]

\[
\therefore f'(-4) = -3(-1)(-5)
\]

\[
\Rightarrow f'(-4) = -15 < 0
\]

\[
f'(x) = -3(x + 3)(x - 1)
\]

\[
\therefore f'(0) = -3(3)(-1)
\]

\[
\Rightarrow f'(0) = 9 > 0
\]

\[
f'(x) = -3(x + 3)(x - 1)
\]

\[
\therefore f'(2) = -3(5)(1)
\]

\[
\Rightarrow f'(2) = -15 < 0
\]

(-3, 25) is a local min. and (1, 7) is a local min.
• Behaviour as $x \to \pm \infty$:

For large values of $x$ (positive or negative), the $x^3$ term is dominant, so that,

$$y = f(x) \approx -x^3$$

As $x \to +\infty$ (i.e. for large and positive $x$),

$f(x) \to -\infty$ (i.e. $y$ is large and negative).

As $x \to -\infty$ (i.e. for large and negative $x$),

$f(x) \to +\infty$ (i.e. $y$ is large and positive).

$y = -x^3 - 3x^2 + 9x + 2$
Example 3

Sketch the graph of the function 
\[ f(x) = 2x^4 - 8x^3. \]

\[ f(x) = 2x^4 - 8x^3 \]
\[ \Rightarrow f(x) = 2x^3(x - 4) \]

* x-intercepts:

\[ f(x) = 0 \]
\[ \Rightarrow 2x^3(x - 4) = 0 \]
\[ \Rightarrow x = 0, x = 4 \]
\[ \therefore (0, 0), (0, 4) \]

* y-intercept:

\[ f(x) = 2x^4 - 8x^3 \]
\[ \therefore f(0) = 2(0)^4 - 8(0)^3 \]
\[ \Rightarrow f(0) = 0 \]
\[ \therefore (0, 0) \]

* SPs:

\[ f'(x) = 0 \]
\[ \therefore 8x^3 - 24x^2 = 0 \]
\[ \Rightarrow 8x^2(x - 3) = 0 \]
\[ \Rightarrow x = 0, x = 3 \]

\[ x = 0: \]
\[ f(x) = 2x^4 - 8x^3 \]
\[ \therefore f(0) = 2(0)^4 - 8(0)^3 \]
\[ \Rightarrow f(0) = 0 \]
\[ \therefore (0, 0) \]

\[ x = 3: \]
\[ f(x) = 2x^4 - 8x^3 \]
\[ \therefore f(3) = 2(3)^4 - 8(3)^3 \]
\[ \Rightarrow f(3) = -54 \]
\[ \therefore (3, -54) \]
\[
f'(x) = 8x^2(x - 3)
\]
\[
\therefore f'(-1) = 8(-4)
\]
\[
\Rightarrow f'(-1) = -32 < 0
\]

\[
f'(x) = 8x^2(x - 3)
\]
\[
\therefore f'(1) = 8(-2)
\]
\[
\Rightarrow f'(1) = -16 < 0
\]

\[
f'(x) = 8x^2(x - 3)
\]
\[
\therefore f'(4) = 128(1)
\]
\[
\Rightarrow f'(4) = 128 > 0
\]

$(0, 0)$ is a P of I and $(3, -54)$ is a local min.
• Behaviour as $x \to \pm \infty$:

For large values of $x$ (positive or negative), the $x^4$ term is dominant, so that,

$$y = f(x) \approx x^4$$

As $x \to +\infty$ (i.e. for large and positive $x$),
$$f(x) \to +\infty$$ (i.e. $y$ is large and negative).

As $x \to -\infty$ (i.e. for large and negative $x$),
$$f(x) \to +\infty$$ (i.e. $y$ is large and positive).

$y = 2x^4 - 8x^3$

(0, 0)

(3, -54)
CfE Higher Maths

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