Area Under a Curve

LI
- Calculate the area under a curve.

SC
- Definite Integration.
Consider the area \( A \) bounded by a curve \( y = f(x) \), the \( x \)-axis and the points \( x = a \) and \( x = b \) (shown below by the shaded area), where the curve is above the \( x \)-axis between \( x = a \) and \( x = b \), but with a point \( c \) between \( a \) and \( b \):

The total area between \( a \) and \( b \) is clearly given by the area between \( a \) and \( c \) plus the area between \( c \) and \( b \):

\[
A = \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]
Example 1 (Non-Calc)

Find the following shaded area:

\[ y = (x - 4)^2 + 2 \]

\[ A = \int_{2}^{5} (x - 4)^2 + 2 \, dx \]

\[ = \int_{2}^{5} (x^2 - 8x + 16) \, dx \]

\[ = \left[ \frac{x^3}{3} - 4x^2 + 18x \right]_{2}^{5} \]

\[ = \left( \frac{125}{3} - 4(5)^2 + 18(5) \right) \]

\[ - \left( \frac{8}{3} - 4(2)^2 + 18(2) \right) \]

\[ = \frac{125}{3} - 100 + 90 + 16 - 36 \]

\[ = 39 - 30 \]

\[ = 9 \text{ square units} \]
Example 2 (Non-Calc)

Find the following shaded area:

\[ y = x^3 - 7x + 6 \]

The values of \( P \) and \( Q \) must first be found:

\[ x^2 - 7x + 6 = 0 \]

\[ (x - 1)(x - 6) = 0 \]

\[ x = 1, 6 \]

So, \( P (1, 0) \) and \( Q (6, 0) \). Hence,

\[ A = - \int_{1}^{6} (x^2 - 7x + 6) \, dx \]

\[ = \left[ \frac{x^3}{3} - \frac{7x^2}{2} + 6x \right]_{1}^{6} \]

\[ = - \left( \frac{6^3}{3} - \frac{7(6)^2}{2} + 6(6) \right) \]

\[ + \left( \frac{1^3}{3} - \frac{7(1)^2}{2} + 6(1) \right) \]

\[ = \frac{1}{3} - \frac{216}{3} + \frac{252}{2} - 36 + 6 \]

\[ = - \frac{215}{3} + \frac{245}{2} - \frac{30}{1} \]

\[ = - \frac{430}{6} + \frac{735}{6} - \frac{180}{6} \]

\[ = \frac{125}{6} \text{ units}^2 \]
The area $A$ bounded by a curve $y = f(x)$, the $x$-axis and the points $x = a$ and $x = b$ (shown below by the shaded area), where the curve is above the $x$-axis between $x = a$ and $x = b$,

is given by the definite integral,

$$A = \int_{a}^{b} f(x) \, dx$$

If the area between the curve $y = g(x) = -f(x)$, the $x$-axis and the points $x = a$ and $x = b$ is considered, the area is obviously the same as before (reflect the above graph in the $x$-axis) and is given by the integral,

$$A = -\int_{a}^{b} g(x) \, dx$$

Thus, when computing an area completely below the $x$-axis, just negative the answer for the definite integral.

Also note that, by looking at the order in which the limits $a$ and $b$ are evaluated,

$$\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx$$
Consider the graph of \( y = \sin x \) between the \( x \)-axis and the points \( x = 0 \) and \( x = 2\pi \). To calculate the area between this curve, the \( x \)-axis and the given points, we must consider the two shaded areas separately (otherwise, the area would be zero - they would cancel out).

The total area is thus given by,

\[
A = \int_{0}^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx
\]
Example 3 (Non-Calc)

Find the area bounded by the curve \( y = 4 \sin x \) and the \( x \)-axis between \( x = \pi/3 \) and \( x = 2\pi \).

Denote the area above the \( x \)-axis by \( A_1 \) and that below the \( x \)-axis by \( A_2 \).

\[
A_1 = \int_{\pi/3}^{\pi} 4 \sin x \, dx
\]

\[
= -4 \left[ \cos x \right]_{\pi/3}^{\pi}
\]

\[
= -4 (\cos \pi - \cos (\pi/3))
\]

\[
= -4 (-1 - 1/2)
\]

\[
= 6
\]

\[
A_2 = -\int_{\pi}^{2\pi} 4 \sin x \, dx
\]

\[
= 4 \left[ \cos x \right]_{\pi}^{2\pi}
\]

\[
= 4 (\cos 2\pi - \cos \pi)
\]

\[
= 4 (1 - (-1))
\]

\[
= 8
\]

Total Area = \( A_1 + A_2 \)

Total Area = 6 + 8

Total Area = 14 \text{ unit}^2
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