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Unit 1 : Differential Equations - Lesson 4

2^{nd}-Order Non-Homogeneous Differential Equations

LI

- Solve DEs of the form $a y'' + b y' + c y = Q(x)$.

SC

- Complementary Function.
- Particular Integral.

A 2^{nd}-order (linear, ordinary) non-homogeneous differential equation (with constant coefficients) is a differential equation that can be written in the form :

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q(x) \quad ☆$$

$$(ay'' + by' + cy = Q(x))$$

$$(a, b, c \in \mathbb{R}, Q(x) \neq 0)$$

Solving the above type of differential equation requires the following steps :

- Solve the associated homogeneous equation :

$$ay'' + by' + cy = 0$$

  The solution to this is called the Complementary Function (CF) y_{CF}.

- Obtain a solution for the equation ☆ . This solution is called the Particular Integral (PI) y_{PI}.

  The form of the PI depends on the form of $Q(x)$.

$Q(x)$	PI
$Cx + D$	$y_{PI} = Rx + T$
$Cx^2 + Dx + E$	$y_{PI} = Rx^2 + Tx + U$
Ce^{px}	$y_{PI} = Re^{px}$
$C\sin px$	$y_{PI} = R\sin px + T\cos px$
$C\cos px$	$y_{PI} = R\sin px + T\cos px$
$Ce^{px} + D$	$y_{PI} = Re^{px} + T$
$C\sin px + D$	$y_{PI} = R\sin px + T\cos px + U$
$C\cos px + D$	$y_{PI} = R\sin px + T\cos px + U$

  The only exception to this is if $Q(x)$ is of the same form as a term in the CF. In which case, take the PI to be $xQ(x)$; if this is of the same form as a term in the CF, take the PI to be $x^2Q(x)$. This process continues.

- The general solution to ☆ is then :

$$\boxed{y_{GS} = y_{CF} + y_{PI}}$$

Aug 24-18:37

Example 1

Obtain the general solution of,

$$y'' - 7y' + 10y = 20x - 4$$ ☆

The Auxiliary Equation is,

$$m^2 - 7m + 10 = 0$$

Solving this for m gives,

$$(m - 2)(m - 5) = 0$$

$$\Rightarrow \qquad \underline{m = 2, m = 5}$$

The CF is thus,

$$\underline{y_{CF} = A e^{2x} + B e^{5x}}$$

For the PI try,

$$y_{PI} = Rx + T$$

$$\therefore \quad y_{PI}' = R$$

$$\Rightarrow \quad y_{PI}'' = 0$$

Substituting this information into ★ gives,

$$0 - 7R + 10(Rx + T) = 20x - 4$$

$$\Rightarrow \quad (10R)x + (10T - 7R) = 20x - 4$$

$$\therefore \quad 10R = 20, \quad 10T - 7R = -4$$

$$\Rightarrow \quad \underline{R = 2, \quad T = 1}$$

$$\therefore \quad \underline{y_{PI} = 2x + 1}$$

$$y_{GS} = y_{CF} + y_{PI}$$

$$\therefore \quad \boxed{y_{GS} = Ae^{2x} + Be^{5x} + 2x + 1}$$

Example 2

Obtain the general solution of,

$$y'' - 4y' + 3y = 6e^{x} \qquad ☆$$

The Auxiliary Equation is,

$$m^2 - 4m + 3 = 0$$

Solving this for m gives,

$$(m - 1)(m - 3) = 0$$

$$\Rightarrow \qquad \underline{m = 1, m = 3}$$

The CF is thus,

$$\underline{y_{CF} = Ae^{x} + Be^{3x}}$$

As the RHS of ★ is of the same form as a term in the CF ($A\,e^{x}$), we try for the PI,

$$y_{PI} = R\,x\,e^{x}$$

$$\therefore \quad y_{PI}' = R\,e^{x} + R\,x\,e^{x}$$

$$\Rightarrow \quad y_{PI}'' = R\,e^{x} + R\,e^{x} + R\,x\,e^{x}$$

$$\Rightarrow \quad y_{PI}'' = 2\,R\,e^{x} + R\,x\,e^{x}$$

Substituting this information into ★ gives,

$$(2\,R\,e^{x} + R\,x\,e^{x}) - 4\,(R\,e^{x} + R\,x\,e^{x}) + 3\,(R\,x\,e^{x}) = 6\,e^{x}$$

$$\Rightarrow \quad -\,2\,R\,e^{x} = 6\,e^{x}$$

$$\Rightarrow \quad \underline{R = -\,3}$$

$$\therefore \quad \underline{y_{PI} = -\,3\,x\,e^{x}}$$

$$y_{GS} = y_{CF} + y_{PI}$$

$$\therefore \quad \boxed{y_{GS} = A\,e^{x} + B\,e^{3x} - 3\,x\,e^{x}}$$

Example 3

Obtain the general solution of,

$$y'' - 2y' + 5y = 30 \sin x \quad \star$$

and also the particular solution satisfying $y(0) = 4$ and $y'(0) = 13$.

The Auxiliary Equation is,

$$m^2 - 2m + 5 = 0$$

Solving this for m gives (check !) $m = 1 \pm 2i$.

The CF is thus,

$$\underline{y_{CF} = e^x (A \cos 2x + B \sin 2x)}$$

For the PI try,

$$y_{PI} = R \sin x + T \cos x$$

$$\therefore \quad y_{PI}' = R \cos x - T \sin x$$

$$\Rightarrow \quad y_{PI}'' = -R \sin x - T \cos x$$

Substituting this information into ★ gives, using the abbreviations $S = \sin x$ and $C = \cos x$,

$$(-RS - TC) - 2(RC - TS) + 5(RS + TC) = 30S$$

$$\Rightarrow \quad (4R + 2T)S + (-2R + 4T)C = 30S$$

$$\therefore \quad 4R + 2T = 30, \quad -2R + 4T = 0$$

$$\Rightarrow \quad \underline{R = 6, \quad T = 3}$$

$$\therefore \quad \underline{y_{PI} = 6 \sin x + 3 \cos x}$$

$$y_{GS} = y_{CF} + y_{PI}$$

$$\therefore \quad \boxed{\begin{aligned} y_{GS} = {} & e^{x}(A \cos 2x + B \sin 2x) \\ & + 6 \sin x + 3 \cos x \end{aligned}}$$

$$y = e^{x}(A\cos 2x + B\sin 2x) + 6\sin x + 3\cos x$$

$$\therefore \quad y' = e^{x}(A\cos 2x + B\sin 2x) + e^{x}(-2A\sin 2x + 2B\cos 2x) + 6\cos x - 3\sin x$$

The initial conditions respectively give,

$$4 = A + 3$$

$$13 = A + 2B + 6$$

$$\therefore \quad \underline{A = 1, \; B = 3}$$

The required particular solution is thus,

$$\boxed{y = e^{x}(\cos 2x + 3\sin 2x) + 6\sin x + 3\cos x}$$

Sep 20-21:40

AH Maths - MiA (2nd Edn.)

- pg. 146 Ex. 8.9 Q 1, 2.

Ex. 8.9

1 Find the general solution to each of these.

a $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 3x + 1$

b $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + 2x + 1$

c $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = e^{\frac{x}{2}}$

d $2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = 50 \sin x$

e $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 50e^{2x}$

f $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 4e^{5x}$

g $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 5x - 3$

h $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 6e^{4x}$

i $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 1$

j $\frac{d^2y}{dx^2} + 4y = 8 \sin 2x$

2 Find the particular solution to each of these differential equations.

a $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 12x + 8$, when $x = 0, y = 2$ and $\frac{dy}{dx} = 6$

b $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$, when $x = 0, y = 3 - e$ and $x = 1, y = e$

c $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = x^2$, when $x = 0, y = 9$ and $\frac{dy}{dx} = 5$

d $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$, when $x = 0, y = -1$ and when $x = 1, y = 4e$

e $\frac{d^2y}{dx^2} + 4y = 6 \cos x$, when $x = 0, y = 3$ and when $x = \frac{\pi}{4}, y = 2 + \sqrt{2}$

Answers to AH Maths (MiA), pg. 146, Ex. 8.9

1 **a** $y = Ae^{x} + Be^{-3x} - x - 1$

b $y = Ae^{2x} + Be^{x} + \frac{1}{2}x^2 + \frac{5}{2}x + \frac{15}{4}$

c $y = Ae^{0.5x} + Be^{-2x} + \frac{1}{5}xe^{0.5x}$

d $y = Ae^{-0.5x} + Be^{-3x} + \sin x - 7\cos x$

e $y = Ae^{-3x} + Bxe^{-3x} + 2e^{2x}$

f $y = Ae^{5x} + Bxe^{5x} + 2x^2e^{5x}$

g $y = Ae^{0.5x} + Bxe^{0.5x} + 5x + 17$

h $y = e^{3x}(A\cos x + B\sin x) + 3e^{4x}$

i $y = e^{2x}(A\cos 2x + B\sin 2x) + \frac{1}{8}$

j $y = (A\cos 2x + B\sin 2x) - 2x\cos 2x$

2 **a** $y = 6e^{3x} - 7e^{2x} + 2x + 3$

b $y = e^{2x} + (2 - e)e^{x} - xe^{x}$

c $y = 3e^{x} - 8e^{0.5x} + x^2 + 6x + 14$

d $y = -e^{x} + 2xe^{x} + 3x^2e^{x}$

e $y = 2\sin 2x + \cos 2x + 2\cos x$