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Linear and Parabolic Motion - Lesson 3

Vector Functions

LI

- Differentiate and integrate vector functions of time.

SC

- Diff. and Int. functions.

Calculus of Vector Functions

$$\boxed{\text{Differentiating a vector means differentiating each component of that vector; similar for integrating a vector}}$$

Reminder

$$\begin{tikzcd}[column sep=huge]
\text{displacement} \arrow[r, bend left, "\frac{d}{dt}"] & \text{velocity} \arrow[r, bend left, "\frac{d}{dt}"] \arrow[l, bend left, "\int"] & \text{acceleration} \arrow[l, bend left, "\int"]
\end{tikzcd}$$

$$\underline{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{v}(t) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\underline{a}(t) = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$$

Example 1

A particle has acceleration $4\,\underline{j} - 2\,\underline{k}$. Initially, it has position vector $-24\,\underline{i} - 72\,\underline{j} + 6\,\underline{k}$ and velocity $4\,\underline{i} + 5\,\underline{k}$. Find :

(a) the velocity at time t.
(b) the speed when $t = 2$.
(c) the position vector at time t.
(d) when the particle passes through the origin.

(a) $\underline{a}(t) = 4\,\underline{j} - 2\,\underline{k}$

$\therefore \quad \underline{v}(t) = 4t\,\underline{j} - 2t\,\underline{k} + \underline{C}$

$\underline{v}(0) = 4\,\underline{i} + 5\,\underline{k}$ gives,

$4\,\underline{i} + 5\,\underline{k} = 4(0)\,\underline{i} - 2(0)\,\underline{k} + \underline{C}$

$\Rightarrow \quad \underline{C} = 4\,\underline{i} + 5\,\underline{k}$

$\therefore \quad \boxed{\underline{v}(t) = 4\,\underline{i} + 4t\,\underline{j} + (5 - 2t)\,\underline{k}}$

(b) $\underline{v}(t) = 4\,\underline{i} + 4t\,\underline{j} + (5 - 2t)\,\underline{k}$

$\therefore \quad \underline{v}(2) = 4\,\underline{i} + 8\,\underline{j} + \underline{k}$

$\therefore \quad \boxed{v(2) = 9\text{ m s}^{-1}}$

(c) $\underline{v}(t) = 4\,\underline{i} + 4t\,\underline{j} + (5 - 2t)\,\underline{k}$

$\therefore \quad \underline{r}(t) = 4t\,\underline{i} + 2t^2\,\underline{j} + (5t - t^2)\,\underline{k} + \underline{D}$

$\underline{r}(0) = -24\,\underline{i} - 72\,\underline{j} + 6\,\underline{k}$ gives (check !),

$\underline{D} = -24\,\underline{i} - 72\,\underline{j} + 6\,\underline{k}$

$$\therefore \quad \boxed{\underline{r}(t) = \begin{pmatrix} 4t - 24 \\ 2t^2 - 72 \\ 6 + 5t - t^2 \end{pmatrix}}$$

(d) Particle passes through origin means,

$\underline{r}(t) = \underline{0}$

$\therefore \quad 4t - 24 = 0 \Rightarrow t = 6$

$2t^2 - 72 = 0 \Rightarrow t^2 = 36 \Rightarrow t = \pm 6$

$6 + 5t - t^2 = 0 \Rightarrow (6 - t)(1 + t) = 0 \Rightarrow t = 6, -1$

As all 3 equations have a common solution ($t = 6$), the particle passes through the origin at $t = 6$ s

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Example 2

Two particles have velocities $2t\underline{i} + 12\underline{j}$ and $4\underline{i} + (3 - 2t)\underline{j}$.

Find when the particles are moving in perpendicular directions and find the displacement of each particle at that instant, assuming both particles start from the origin.

$$\underline{v}_A(t) = 2t\underline{i} + 12\underline{j} \quad , \quad \underline{v}_B(t) = 4\underline{i} + (3 - 2t)\underline{j}$$

Particles moving perpendicularly means that their velocity vectors are perpendicular (so their scalar product vanishes),

$$\underline{v}_A(t) \bullet \underline{v}_B(t) = 0$$

$$\therefore \quad (2t\underline{i} + 12\underline{j}) \bullet (4\underline{i} + (3 - 2t)\underline{j}) = 0$$

$$\Rightarrow \quad 8t + 12(3 - 2t) = 0$$

$$\Rightarrow \quad 8t + 36 - 24t = 0$$

$$\Rightarrow \quad 16t = 36$$

$$\Rightarrow \quad \boxed{t = 9/4\text{ s}}$$

$$\underline{v}_A(t) = 2t\underline{i} + 12\underline{j}$$

$$\therefore \quad \underline{r}_A(t) = t^2\underline{i} + 12t\underline{j} + \underline{C}$$

$$\underline{r}_A(0) = \underline{0} \Rightarrow \underline{\underline{C} = \underline{0}}$$

$$\therefore \quad \underline{\underline{r}_A(t) = t^2\underline{i} + 12t\underline{j}}$$

$$\therefore \quad \boxed{\underline{r}_A(9/4) = (81/16)\underline{i} + 27\underline{j}}$$

$$\underline{v}_B(t) = 4\underline{i} + (3 - 2t)\underline{j}$$

$$\therefore \quad \underline{r}_B(t) = 4t\underline{i} + (3t - t^2)\underline{j} + \underline{D}$$

$$\underline{r}_B(0) = \underline{0} \Rightarrow \underline{\underline{D} = \underline{0}}$$

$$\therefore \quad \underline{\underline{r}_B(t) = 4t\underline{i} + (3t - t^2)\underline{j}}$$

$$\therefore \quad \boxed{\underline{r}_B(9/4) = 9\underline{i} + (27/16)\underline{j}}$$

Example 3

A particle has position vector $\underline{\mathbf{r}}(t) = 2\cos(17t)\,\underline{\mathbf{i}} - 2\sin(17t)\,\underline{\mathbf{k}}$.

Show that the particle moves in a circle, stating the radius.

Show also that the acceleration vector is proportional to the displacement, stating the proportionality constant.

$$\underline{\mathbf{r}}(t) = 2\cos(17t)\,\underline{\mathbf{i}} - 2\sin(17t)\,\underline{\mathbf{k}} = x(t)\,\underline{\mathbf{i}} + z(t)\,\underline{\mathbf{k}}$$

$$\therefore \quad x(t) = 2\cos(17t) \ , \quad z(t) = -2\sin(17t)$$

$$\therefore \quad (x(t))^2 + (z(t))^2 = 4\cos^2(17t) + 4\sin^2(17t)$$

$$\Rightarrow \quad (x(t))^2 + (z(t))^2 = 4(\cos^2(17t) + \sin^2(17t))$$

$$\Rightarrow \quad \underline{(x(t))^2 + (z(t))^2 = 4}$$

$x^2 + z^2 = 4$ is a circle in the x - z plane with centre $(0, 0)$ and radius 2

$$\underline{\mathbf{r}}(t) = 2\cos(17t)\,\underline{\mathbf{i}} - 2\sin(17t)\,\underline{\mathbf{k}}$$

$$\therefore \quad \underline{\mathbf{v}}(t) = -2(17)\sin(17t)\,\underline{\mathbf{i}} - 2(17)\cos(17t)\,\underline{\mathbf{k}}$$

$$\therefore \quad \underline{\mathbf{a}}(t) = -2(17)^2\cos(17t)\,\underline{\mathbf{i}} + 2(17)^2\sin(17t)\,\underline{\mathbf{k}}$$

$$\Rightarrow \quad \underline{\mathbf{a}}(t) = -289\,(2\cos(17t)\,\underline{\mathbf{i}} - 2\sin(17t)\,\underline{\mathbf{k}})$$

$$\Rightarrow \quad \underline{\underline{\mathbf{a}}(t) = -289\,\underline{\mathbf{r}}(t)}$$

As, $\underline{\mathbf{a}}(t) = -289\,\underline{\mathbf{r}}(t)$, $\underline{\mathbf{a}}(t) \propto \underline{\mathbf{r}}(t)$, with the proportionality constant being -289

Blue Book

- pg. 23 Ex. 2 B Q 2, 9, 10, 12, 16.
- pg. 398-399 Ex. 16 A Q 15 - 26, 33 - 43.