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Differentiation and Properties of Functions - Lesson 3

Stationary Points

LI

- Know the different types of Stationary Points (SPs).
- Determine the coordinates and nature of SPs of functions.

SC

- Differentiate.
- Nature Table.

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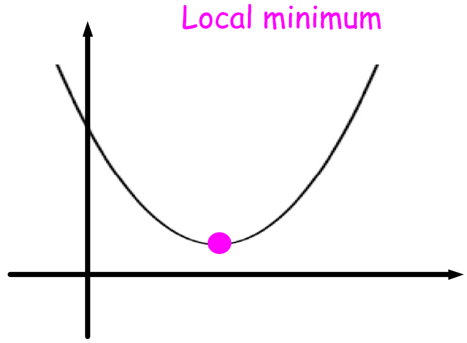
A Stationary Point (SP) is a point where a function has zero derivative

Types (aka Nature) of Stationary Points

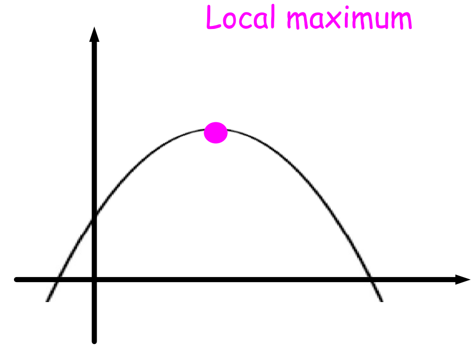
TURNING POINTS

Local minimum



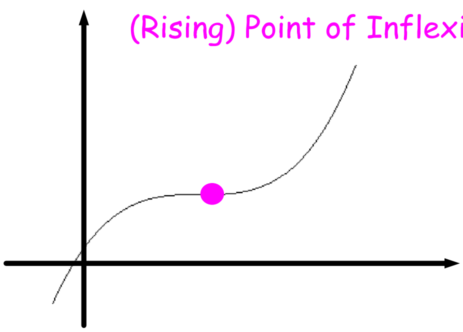
Gradient goes from – ve to 0 to + ve

Local maximum



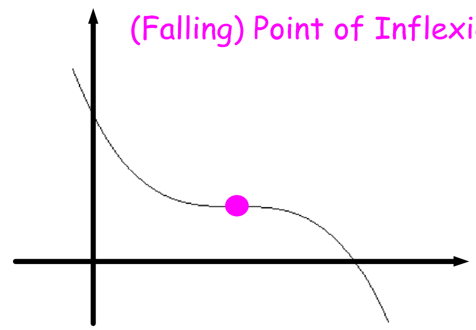
Gradient goes from + ve to 0 to – ve

(Rising) Point of Inflexion



Gradient goes from + ve to 0 to + ve

(Falling) Point of Inflexion



Gradient goes from – ve to 0 to – ve

Strategy for finding SPs

- Differentiate function.
- Put derivative equal to 0 and solve for x.
- Get y - value(s) from original function.
- Draw Nature Table.
- State conclusion.

Example 1

Determine the coordinates and nature of the stationary points of the function

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 12x + 5.$$

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 12x + 5$$

$$\therefore \quad f'(x) = x^2 - 4x - 12$$

For stationary points (SPs), $f'(x) = 0$:

$$x^2 - 4x - 12 = 0$$

$$\therefore \quad (x - 6)(x + 2) = 0$$

$$\Rightarrow \quad \underline{x = 6, x = -2}$$

$\underline{x = 6:}$

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 12x + 5$$

$$\therefore \quad f(6) = \frac{1}{3}(6)^3 - 2(6)^2 - 12(6) + 5$$

$$\Rightarrow \quad f(6) = 72 - 72 - 72 + 5$$

$$\Rightarrow \quad \underline{f(6) = -67}$$

$\underline{x = -2:}$

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 12x + 5$$

$$\therefore \quad f(-2) = \frac{1}{3}(-2)^3 - 2(-2)^2 - 12(-2) + 5$$

$$\Rightarrow \quad f(-2) = -\frac{8}{3} - 8 + 24 + 5$$

$$\Rightarrow \quad \underline{f(-2) = \frac{55}{3}}$$

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x	$\xrightarrow{-3}$	-2	$\xrightarrow{0}$	6	$\xrightarrow{7}$
$f'(x)$	$+$	0	$-$	0	$+$
Slope	/	—	\	—	/

$$f'(x) = (x - 6)(x + 2)$$

$$\therefore \quad f'(-3) = (-3 - 6)(-3 + 2)$$

$$\Rightarrow \quad \underline{f'(-3) = 9 > 0}$$

$$f'(x) = (x - 6)(x + 2)$$

$$\therefore \quad f'(0) = (0 - 6)(0 + 2)$$

$$\Rightarrow \quad \underline{f'(0) = -12 < 0}$$

$$f'(x) = (x - 6)(x + 2)$$

$$\therefore \quad f'(7) = (7 - 6)(7 + 2)$$

$$\Rightarrow \quad \underline{f'(7) = 9 > 0}$$

$\left(-2, \dfrac{55}{3}\right)$ is a local max,

and $(6, -67)$ is a local min.

Example 2

Find the coordinates and nature of the stationary points of the function

$$f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + 3.$$

$$f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + 3$$

$$\therefore \quad f'(x) = x^3 - 4x^2 + 4x$$

For SPs, $f'(x) = 0$:

$$x^3 - 4x^2 + 4x = 0$$

$$\therefore \quad x(x^2 - 4x + 4) = 0$$

$$\Rightarrow \quad x(x - 2)(x - 2) = 0$$

$$\Rightarrow \quad \underline{x = 0, x = 2}$$

$\underline{x = 0:}$

$$f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + 3$$

$$\therefore \quad f(0) = \frac{1}{4}(0)^4 - \frac{4}{3}(0)^3 + 2(0)^2 + 3$$

$$\Rightarrow \quad \underline{f(0) = 3}$$

$\underline{x = 2:}$

$$f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + 3$$

$$\therefore \quad f(2) = \frac{1}{4}(2)^4 - \frac{4}{3}(2)^3 + 2(2)^2 + 3$$

$$\Rightarrow \quad f(2) = 4 - \frac{32}{3} + 8 + 3$$

$$\Rightarrow \quad \underline{f(2) = \frac{13}{3}}$$

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x	$\xrightarrow{-1}$	0	$\xrightarrow{1}$	2	$\xrightarrow{3}$
f ' (x)	−	0	+	0	+
Slope	\	—	/	—	/

$$f'(x) = x(x-2)^2$$

$$\therefore \quad f'(-1) = -1(-1-2)^2$$

$$\Rightarrow \quad \underline{f'(-1) = -9 < 0}$$

$$f'(x) = x(x-2)^2$$

$$\therefore \quad f'(1) = 1(1-2)^2$$

$$\Rightarrow \quad \underline{f'(1) = 1 > 0}$$

$$f'(x) = x(x-2)^2$$

$$\therefore \quad f'(3) = 3(3-2)^2$$

$$\Rightarrow \quad \underline{f'(3) = 3 > 0}$$

(0, 3) is a local min. and

$\left(2, \frac{13}{3}\right)$ is a (rising) P of I

Example 3

Show that the graph of the cubic curve $y = x^3 + 2x^2 + 2x + 7$ has no horizontal tangent.

A horizontal tangent means that there is at least one point on the graph at which the gradient is 0. This means that the derivative would have to be 0 there.

$$y = x^3 + 2x^2 + 2x + 7$$

$$\therefore \quad y' = 3x^2 + 4x + 2$$

For a horizontal tangent, there must be solutions for x to the equation $y' = 0$:

$$3x^2 + 4x + 2 = 0$$

The discriminant of this quadratic is $16 - 24 = -8$. As the discriminant is < 0, there are no solutions to $y' = 0$.

As there are no solutions to $y' = 0$, there is no horizontal tangent to $y = x^3 + 2x^2 + 2x + 7$

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Example 4

Show that the function defined by
$f(x) = -x^3 - 3x^2 - 3x - 1$ has exactly one stationary point and determine its coordinates and nature.

$$f(x) = -x^3 - 3x^2 - 3x - 1$$

$$\therefore \quad f'(x) = -3x^2 - 6x - 3$$

For SPs, $f'(x) = 0$:

$$-3x^2 - 6x - 3 = 0$$

$$\therefore \quad -3(x^2 + 2x + 1) = 0$$

$$\Rightarrow \quad -3(x + 1)^2 = 0$$

$$\Rightarrow \quad \underline{x = -1}$$

As, there is only 1 solution to $f'(x) = 0$,
$f(x) = x^3 + 3x^2 + 3x + 1$ has exactly 1 SP

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$$f(x) = -x^3 - 3x^2 - 3x - 1$$

$$\therefore \quad f(-1) = -(-1)^3 - 3(-1)^2 - 3(-1) - 1$$

$$\Rightarrow \quad f(-1) = 1 - 3 + 3 - 1$$

$$\Rightarrow \quad \underline{f(-1) = 0}$$

x	$\xrightarrow{-2}$	-1	$\xrightarrow{0}$
$f'(x)$	$-$	0	$-$
Slope	$\searrow$	—	$\searrow$

$$f'(x) = -3(x+1)^2$$

$$\therefore \quad f'(-2) = -3(-2+1)^2$$

$$\Rightarrow \quad \underline{f'(-2) = -3 < 0}$$

$$f'(x) = -3(x+1)^2$$

$$\therefore \quad f'(0) = -3(0+1)^2$$

$$\Rightarrow \quad \underline{f'(0) = -3 < 0}$$

$$\boxed{(-1, 0) \text{ is a (falling) P of I}}$$

Example 5

Find the coordinates and nature of the stationary points of $y = \cos x$ $(0 \leq x \leq 2\pi)$.

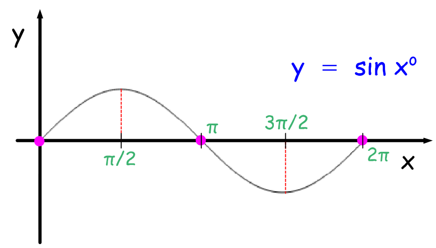
$$y(x) = \cos x$$

$$\therefore \quad y'(x) = -\sin x$$

For SPs, $f'(x) = 0$:

$$-\sin x = 0$$

$$\therefore \quad \sin x = 0$$



$$\therefore \quad \underline{x = 0, \pi, 2\pi}$$

$\underline{x = 0:}$

$$y(x) = \cos 0$$

$$\therefore \quad y(0) = \cos 0$$

$$\Rightarrow \quad \underline{y(0) = 1}$$

$\underline{x = \pi:}$

$$y(x) = \cos x$$

$$\therefore \quad y(\pi) = \cos \pi$$

$$\Rightarrow \quad \underline{y(\pi) = -1}$$

$\underline{x = 2\pi:}$

$$y(x) = \cos x$$

$$\therefore \quad y(2\pi) = \cos 2\pi$$

$$\Rightarrow \quad \underline{y(2\pi) = 1}$$

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x	$\xrightarrow{-1}$	0	$\xrightarrow{1}$	π	$\xrightarrow{4}$	2π	$\xrightarrow{7}$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
Slope	/	—	\	—	/	—	\

$$y'(x) = -\sin x$$
$$\therefore \quad y'(-1) = -\sin(-1)$$
$$\Rightarrow \quad \underline{y'(-1) = 0.841\ldots > 0}$$

$$y'(x) = -\sin x$$
$$\therefore \quad y'(1) = -\sin(1)$$
$$\Rightarrow \quad \underline{y'(1) = -0.841\ldots < 0}$$

$$y'(x) = -\sin x$$
$$\therefore \quad y'(4) = -\sin(4)$$
$$\Rightarrow \quad \underline{y'(4) = 0.756\ldots > 0}$$

$$y'(x) = -\sin x$$
$$\therefore \quad y'(7) = -\sin(7)$$
$$\Rightarrow \quad \underline{y'(7) = -0.656\ldots < 0}$$

$(0, 1)$ is a local max., $(\pi, -1)$ is a local min. and $(2\pi, 1)$ is a local max.

Example 6

Find the x - coordinates of the stationary points of

$R(x) = x + \frac{9}{x}$ ($x \neq 0$).

$$R(x) = x + \frac{9}{x}$$

$$R(x) = x + 9x^{-1}$$

$$\therefore \quad R'(x) = 1 - 9x^{-2}$$

For SPs, $R'(x) = 0$:

$$1 - 9x^{-2} = 0$$

$$1 - \frac{9}{x^2} = 0$$

$$\Rightarrow \quad \frac{9}{x^2} = 1$$

$$\Rightarrow \quad x^2 = 9$$

$$\Rightarrow \quad \boxed{x = \pm 3}$$

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