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Applications of Calculus - Lesson 3

Rates of Change

LI

- Solve problems involving rates of change.

SC

- Chain Rule.

Differentiation is the study of rates of change; how quickly one variable changes with respect to another.

Usually, especially in real-life applications (in particular, physics), rates of change occur with respect to time.

Connections Between Displacement, Velocity and Acceleration

$$
\begin{tikzcd}
s(t) \arrow[r, shift left, "\frac{d}{dt}"] & v(t) \arrow[l, shift left, "\int"] \arrow[r, shift left, "\frac{d}{dt}"] & a(t) \arrow[l, shift left, "\int"]
\end{tikzcd}
$$

More specifically :

$$\frac{d}{dt}\, s(t) = v(t) \longleftrightarrow \int v(t)\, dt = s(t) + C$$

(derivative of displacement is velocity) (integral of velocity is displacement)

$$\frac{d}{dt}\, v(t) = a(t) \longleftrightarrow \int a(t)\, dt = v(t) + D$$

(derivative of velocity is acceleration) (integral of acceleration is velocity)

We will be interested in the derivative equations in this lesson.

Chain Rule with Independent Variable being Time

The Chain Rule for a function $y = f(g(t))$, writing $u = g(t)$, is,

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$$

Some important special cases of the Chain Rule include :

$$\frac{d}{dt}(y(t))^2 = 2y(t)\frac{dy}{dt}$$

$$\frac{d}{dt}(y(t))^3 = 3(y(t))^2\frac{dy}{dt}$$

Example 1

Relative to a suitable origin a particle moves along the x - axis such that its displacement, s metres, at time t seconds, is given by,

$$s(t) = t^3 - 7t^2 + 8t + 5$$

Find :

(a) how far from the origin the particle is at $t = 0$.
(b) the velocity of the particle when $t = 2$.
(c) when the particle has zero velocity.

(a)

$$s(t) = t^3 - 7t^2 + 8t + 5$$

$$\therefore \quad s(0) = (0)^3 - 7(0) + 8(0) + 5$$

$$\Rightarrow \quad \boxed{s(0) = 5 \text{ m}}$$

(b)

$$v(t) = s'(t)$$

$$\therefore \quad v(t) = \frac{d}{dt}(t^3 - 7t^2 + 8t + 5)$$

$$\Rightarrow \quad v(t) = 3t^2 - 14t + 8$$

$$\therefore \quad v(2) = 3(2)^2 - 14(2) + 8$$

$$\Rightarrow \quad v(2) = 12 - 28 + 8$$

$$\Rightarrow \quad \boxed{v(2) = -8 \text{ m/s}}$$

(c) Particle has zero velocity when $v(t) = 0$:

$$v(t) = 0$$

$$\therefore \quad 0 = 3t^2 - 14t + 8$$

$$\Rightarrow \quad 0 = (3t - 2)(t - 4)$$

$$\Rightarrow \quad \boxed{t = 2/3 \text{ s}, 4 \text{ s}}$$

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Example 2

The volume of a sphere is increasing at a rate of 120 cm ^{3}.

Find how fast the radius is increasing when the radius is 4 cm.

The volume of a sphere is,

$$V = \frac{4\pi}{3} r^3$$

As the radius changes over time, so will the volume; so, they are both functions of time :

$$V(t) = \frac{4\pi}{3} (r(t))^3$$

'Rate' means 'derivative'. So,

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi}{3}(r(t))^3\right)$$

$$\frac{dV}{dt} = \frac{4\pi}{3}\frac{d}{dt} r^3$$

$$\frac{dV}{dt} = \frac{4\pi}{3}\left(3r^2\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

We have that $\frac{dV}{dt} = 120$ cm ^{3} when $r = 4$ cm :

We need to solve for $\frac{dr}{dt}$.

$$4\pi(4)^2\frac{dr}{dt} = 120$$

$$64\pi\frac{dr}{dt} = 120$$

$$\frac{dr}{dt} = \frac{120}{64\pi}$$

$$\boxed{\frac{dr}{dt} = \frac{15}{8\pi} \text{ cm/s}}$$

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