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Further Proof Techniques - Lesson 3

Proof by Contraposition and Proof by Contradiction

LI

- Know the contrapositive of a statement.
- Prove statements using Proof by Contraposition.
- Prove statements using Proof by Contradiction.

SC

- Logical reasoning.

Jan 10-12:11

The negation of A is 'not A'

The contrapositive of the statement $A \Rightarrow B$ is the statement 'not B' $\Rightarrow$ 'not A'

The contrapositive of a statement is logically equivalent to the original statement; thus, to prove $A \Rightarrow B$, we can prove 'not B' $\Rightarrow$ 'not A' (often easier)

Proof by Contraposition (aka Proof by Contrapositive) is a proof technique that involves proving $A \Rightarrow B$ by proving 'not B' $\Rightarrow$ 'not A'

Proof by Contradiction is a proof technique that involves proving $A \Rightarrow B$ by assuming A and 'not B', then reaching a contradiction (usually, 'not A')

Notation

$\mathbb{N}$ - set of all natural numbers.

$\mathbb{W}$ - set of all whole numbers.

$\mathbb{Z}$ - set of all integers.

$\mathbb{Q}$ - set of all rational numbers.

$\mathbb{R}$ - set of all real numbers (all rational and irrational numbers).

$\mathbb{C}$ - set of all complex numbers.

Feb 21-22:34

Example 1

Prove by contraposition that : n^2 odd $\Rightarrow$ n odd ($n \in \mathbb{Z}$).

Assume the negation of 'n odd'; i.e. assume that n is even.
Then $\exists\, k \in \mathbb{Z}$ such that $n = 2k$. Hence,

$$n^2 = (2k)^2$$

$$\Rightarrow \quad n^2 = 4k^2$$

$$\Rightarrow \quad \underline{n^2 = 2(2k^2)}$$

As $k \in \mathbb{Z}$, $2k^2 \in \mathbb{Z}$; hence n^2 is even. So, n even $\Rightarrow$ n^2 even.

$\boxed{\text{By contraposition, } n^2 \text{ odd} \Rightarrow n \text{ odd}}$

Example 2

Prove by contraposition : $x^2 - 6x + 5$ even $\Rightarrow$ x odd ($x \in \mathbb{Z}$).

Assume the negation of 'x odd'; i.e. assume that x is even.

Then $\exists\, k \in \mathbb{Z}$ such that $x = 2k$. Hence,

$$x^2 - 6x + 5 = (2k)^2 - 6(2k) + 5$$

$$\Rightarrow \quad x^2 - 6x + 5 = 4k^2 - 12k + 4 + 1$$

$$\Rightarrow \quad x^2 - 6x + 5 = 2(2k^2 - 6k + 2) + 1$$

As $k \in \mathbb{Z}$, $2k^2 - 6k + 2 \in \mathbb{Z}$; hence $x^2 - 6x + 5$ is odd.

So, x even $\Rightarrow$ $x^2 - 6x + 5$ odd.

$$\boxed{\text{By contraposition, } x^2 - 6x + 5 \text{ even} \Rightarrow x \text{ odd}}$$

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Example 3

Prove by contradiction : $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational. Then $\exists\, a, b \in \mathbb{Z}$ $(b \neq 0)$ such that,

$$\sqrt{2} = \frac{a}{b}$$

where it can be assumed that a and b have no common factors; if they did, they can be cancelled out. Then,

$$a^2 = 2b^2 \Rightarrow a^2 \text{ is even} \Rightarrow \underline{a \text{ is even}}$$

$$\therefore \quad a = 2k \ (k \in \mathbb{Z})$$

$$\Rightarrow \quad 4k^2 = 2b^2$$

$$\Rightarrow \quad b^2 = 2k^2 \Rightarrow b^2 \text{ is even} \Rightarrow \underline{b \text{ is even}}$$

Hence, both a and b are even; but this contradicts the fact that a and b were assumed to have no common factors. Hence, the starting assumption that $\sqrt{2}$ is rational is false.

$$\boxed{\text{By contradiction, } \sqrt{2} \text{ is irrational}}$$

Example 4

Prove by contradiction : x irrational $\Rightarrow$ $x + 3$ irrational.

Assume that $x + 3$ is rational. Then $\exists\, a, b \in \mathbb{Z}$ $(b \neq 0)$ such that,

$$x + 3 = \frac{a}{b}$$

$$\therefore \quad x = \frac{a}{b} - 3$$

$$\Rightarrow \quad x = \frac{a - 3b}{b}$$

As $a, b \in \mathbb{Z}$ $(b \neq 0)$, $a - 3b \in \mathbb{Z}$; hence, $x \in \mathbb{Q}$, contradicting the assumption that x is irrational.

$\boxed{\text{By contradiction, } x \text{ irrational} \Rightarrow x + 3 \text{ irrational}}$

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Example 5

Prove by contradiction : $4 + 7\sqrt{3}$ is irrational. You may assume that $\sqrt{3}$ is irrational.

Assume that $4 + 7\sqrt{3}$ is rational. Then $\exists\, a, b \in \mathbb{Z}$ $(b \neq 0)$ such that,

$$4 + 7\sqrt{3} = \frac{a}{b}$$

$$\therefore \quad 7\sqrt{3} = \frac{a}{b} - 4$$

$$\Rightarrow \quad 7\sqrt{3} = \frac{a - 4b}{b}$$

$$\Rightarrow \quad \sqrt{3} = \frac{a - 4b}{7b}$$

As $a, b \in \mathbb{Z}$, $a - 4b, 7b \in \mathbb{Z}$ $(7b \neq 0)$; hence, $\sqrt{3} \in \mathbb{Q}$, contradicting the assumption that $\sqrt{3}$ is irrational.

$\boxed{\text{By contradiction, } 4 + 7\sqrt{3} \text{ is irrational}}$

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Questions

Prove by contraposition :

1) n^2 even $\Rightarrow$ n even ($n \in \mathbb{Z}$).

2) $x^3 + 1$ even $\Rightarrow$ x odd ($x \in \mathbb{Z}$).

3) $x^3 - 3$ even $\Rightarrow$ x odd ($x \in \mathbb{Z}$).

4) $x^2 + 5x < 0 \Rightarrow x < 0$ ($x \in \mathbb{R}$).

Prove by contradiction :

5) n^2 odd $\Rightarrow$ n odd ($n \in \mathbb{Z}$).

6) n^2 even $\Rightarrow$ n even ($n \in \mathbb{Z}$).

7) x irrational $\Rightarrow$ $x - 8$ irrational.

8) $\sqrt{7}$ is irrational.

9) $13 - 9\sqrt{7}$ is irrational (assume that $\sqrt{7}$ is irrational).

Nov 24-00:35