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Unit 2 : Proof by Mathematical Induction - Lesson 3

Mathematical Induction 3 (Inequalities)

LI

- Use Proof by Mathematical Induction to solve problems involving inequalities.

SC

- Algebra.

Proof by Mathematical Induction

The Principle of Mathematical Induction (PMI) states that, to prove a statement (P (n)) about an infinite set of natural numbers :

- Prove the Base Case : $P(n_0)$ is true.
- Prove the Inductive Step :

  $P(k)$ true $\Rightarrow$ $P(k+1)$ true.

Then $P(n)$ is true $\forall\, n \geq n_0$.

Usually, $n_0 = 1$; then we would state the conclusion as '$P(n)$ is true $\forall\, n \in \mathbb{N}$'.

Example 1

Prove by induction that $n! > 5^n$ for all $n \geq 12$.

$$P(n): \ n! > 5^n$$

Base Case

$$\text{LHS} = 12! = 479\,001\,600$$

$$\text{RHS} = 5^{12} = 244\,140\,625$$

$$\boxed{\text{As LHS} > \text{RHS}, P(12) \text{ is true}}$$

Inductive Step

Assume $P(k)$ is true for some natural number $k \geq 12$, i.e.
assume that :

$$k! > 5^k \longleftarrow$$ Inductive Hypothesis

RTP statement

$$(k + 1)! > 5^{k+1}$$

$$(k + 1)! = (k + 1) \,.\, k!$$

$$> (k + 1) \,.\, 5^k$$

$$> 5 \,.\, 5^k$$

$k \geq 12$
$\Rightarrow k + 1 \geq 13 > 5$

$$\therefore \quad (k + 1)! > 5^{k+1}$$

$$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k + 1) \text{ true}}$$

'$P(12)$ true' and '$P(k)$ true $\Rightarrow P(k + 1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall\, n \geq 12$

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Example 2

Prove by induction that $n^2 > 2n$ ($\forall n \geq 3$).

$$P(n): n^2 > 2n$$

Base Case

$$LHS = 3^2 = 9$$

$$RHS = 2(3) = 6$$

$$\boxed{\text{As LHS} > \text{RHS}, P(3) \text{ is true}}$$

Inductive Step

Assume $P(k)$ is true for some natural number $k \geq 3$, i.e. assume that :

$$k^2 > 2k \longleftarrow$$ Inductive Hypothesis

RTP statement

$$(k+1)^2 > 2(k+1)$$

$$(k+1)^2 = (k+1).(k+1)$$

$$= k^2 + 2k + 1$$

$$> 1 + 2k + 1$$

$k \geq 3$
$\Rightarrow k^2 \geq 9 > 1$

$$= 2k + 2$$

$$\therefore \quad (k+1)^2 > 2(k+1)$$

$$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true}}$$

'$P(3)$ true' and '$P(k)$ true $\Rightarrow P(k+1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall n \geq 3$

Example 3

Prove by induction that $n! > n^2$ $(\forall n \geq 4)$.

$$P(n): n! > n^2$$

Base Case

$$LHS = 4! = 24$$

$$RHS = 4^2 = 16$$

$\boxed{\text{As LHS} > \text{RHS}, P(4) \text{ is true}}$

Inductive Step

Assume $P(k)$ is true for some natural number $k \geq 4$, i.e. assume that :

$$k! > k^2 \longleftarrow$$ Inductive Hypothesis

RTP statement

$$(k+1)! > (k+1)^2$$

$$\begin{aligned}(k+1)! &= (k+1).k! \\ &> (k+1).k^2 \\ &> (k+1).(k+1)\end{aligned}$$

$$\begin{aligned}&k \geq 4 \\ \Rightarrow\ &k^2 \geq 4k \\ &\quad = 2k + 2k \\ &\quad > k + 1\end{aligned}$$

$$\therefore \quad (k+1)! > (k+1)^2$$

$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true}}$

'P (4) true' and 'P (k) true ⇒ P (k + 1) true' together imply, by the PMI, that $P(n)$ is true $\forall n \geq 4$

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Example 4

Prove by induction that $2^n > n^2$ ($\forall n \geq 5$).

$$P(n): 2^n > n^2$$

Base Case

$$LHS = 2^5 = 32$$

$$RHS = 5^2 = 25$$

As LHS > RHS, P (5) is true

Inductive Step

Assume P (k) is true for some natural number $k \geq 5$, i.e. assume that :

$$2^k > k^2 \longleftarrow \text{Inductive Hypothesis}$$

RTP statement

$$2^{k+1} > (k+1)^2$$

$$\begin{aligned} 2^{k+1} &= 2^k . 2 \\ &> 2k^2 \\ &= k^2 + k^2 \\ &\geq k^2 + 5k \\ &= k^2 + 2k + 3k \\ &> k^2 + 2k + 1 \end{aligned}$$

$k \geq 5$
$\Rightarrow k^2 \geq 5k$

$$\therefore \quad 2^{k+1} > (k+1)^2$$

Hence, P (k) true $\Rightarrow$ P (k + 1) true

'P (5) true' and 'P (k) true $\Rightarrow$ P (k + 1) true' together imply, by the PMI, that P (n) is true $\forall n \geq 5$

Questions (and 'Answers' !)

Prove by mathematical induction that :

1) $n^2 > 4n$ $(\forall n \geq 5)$.

2) $n! > 3^n$ $(\forall n \geq 7)$.

3) $6n + 6 < 2^n$ $(\forall n \geq 6)$.

4) $3^n < (n + 1)!$ $(\forall n \geq 4)$.

5) $n! \leq n^n$ $(\forall n \in \mathbb{N})$.

6) $n! > n^3$ $(\forall n \geq 6)$.