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Unit 1 : Integral Calculus - Lesson 3

Integration of Rational Functions

LI

- Integrate rational functions.

SC

- Partial fractions.
- Standard integrals.

Integrating a rational function involves rewriting the rational function using partial fractions

The integrated function normally gives answers involving natural logarithms or inverse tangent

Example 1

Integrate $\frac{5x + 11}{(x + 1)(x + 3)}$, writing the answer in the form $\ln |(x + 1)^m(x + 3)^n| + C$, stating the values of m and n.

$$\text{Let } I = \int \frac{5x + 11}{(x + 1)(x + 3)} dx.$$

Performing partial fractions gives (check !),

$$\frac{5x + 11}{(x + 1)(x + 3)} = \frac{3}{x + 1} + \frac{2}{x + 3}$$

$$I = \int \frac{5x + 11}{(x + 1)(x + 3)} dx$$

$$\therefore I = \int \frac{3}{x + 1} dx + \int \frac{2}{x + 3} dx$$

$$\Rightarrow I = 3 \ln |x + 1| + 2 \ln |x + 3| + C$$

$$\Rightarrow I = \ln |x + 1|^3 + \ln |x + 3|^2 + C$$

$$\Rightarrow I = \ln |(x + 1)^3(x + 3)^2| + C$$

$(m = 3, n = 2)$

Example 2

Integrate $\frac{x^2 - 4x + 13}{(x + 1)(x - 2)^2}$.

$$\text{Let } I = \int \frac{x^2 - 4x + 13}{(x + 1)(x - 2)^2} dx.$$

Performing partial fractions gives (check !),

$$\frac{x^2 - 4x + 13}{(x + 1)(x - 2)^2} = \frac{2}{x + 1} - \frac{1}{x - 2} + \frac{3}{(x - 2)^2}$$

$$I = \int \frac{x^2 - 4x + 13}{(x + 1)(x - 2)^2} dx$$

$$\therefore I = \int \frac{2}{x + 1} dx - \int \frac{1}{x - 2} dx + \int \frac{3}{(x - 2)^2} dx$$

$$\Rightarrow I = 2 \ln|x + 1| - \ln|x - 2| - 3(x - 2)^{-1} + C$$

Example 3

Integrate $\frac{5x^2 - 2x + 7}{(x - 1)(x^2 + 4)}$.

$$\text{Let } I = \int \frac{5x^2 - 2x + 7}{(x - 1)(x^2 + 4)} dx.$$

Performing partial fractions gives (check !),

$$\frac{5x^2 - 2x + 7}{(x - 1)(x^2 + 4)} = \frac{2}{x - 1} + \frac{3x + 1}{x^2 + 4}$$

$$I = \int \frac{5x^2 - 2x + 7}{(x - 1)(x^2 + 4)} dx$$

$$\therefore I = \int \frac{2}{x - 1} dx + \int \frac{3x + 1}{x^2 + 4} dx$$

$$\Rightarrow I = \int \frac{2}{x - 1} dx + 3 \int \frac{x}{x^2 + 4} dx + \int \frac{1}{x^2 + 4} dx$$

$$\Rightarrow I = 2 \ln|x - 1| + (3/2) \ln|x^2 + 4| + (1/2) \tan^{-1}(x/2) + C$$

AH Maths - MiA (2nd Edn.)

- pg. 113-4 Ex. 7.7 Q 1c - e, 2c, d, 3c, d, 4b, 5, 6c, e, 7a - c, 8a, b.

Ex. 7.7

- 1** Integrate each function by first resolving it into partial fractions.

c) $\frac{2x - 7}{(x - 2)(x - 3)}$

d) $\frac{13x + 3}{(3x + 1)(x - 1)}$

e) $\frac{1}{(x - 1)(x + 2)}$

- 2** Integrate each function by first resolving it into partial fractions.

You will have to factorise the denominator first.

c) $\frac{5x - 2}{x^2 - 3x + 2}$

d) $\frac{x + 9}{x^2 - 9}$

- 3** Integrate each function by first resolving it into partial fractions.

c) $\frac{x^2 - 7x + 8}{x(x - 2)^2}$

d) $\frac{1}{(x - 1)(x - 2)^2}$

- 4** Integrate each function by first resolving it into partial fractions.

b) $\frac{x}{x^3 + 3x^2 - 4}$

- 5** Find

a) $\int \frac{2x^2 + 1}{(x + 1)(x^2 + 2)} dx$

b) $\int \frac{5x^2 + 3}{x(x^2 + 1)} dx$

c) $\int \frac{3x^2 + 5}{(x - 1)(x^2 + 3)} dx$

d) $\int \frac{1}{x(x^2 + 5)} dx$

e) $\int \frac{x}{(x - 2)(x^2 + 3)} dx$

- 6** Perform a division, then integrate.

c) $\int \frac{2x^3 + 3x^2 - 3x + 2}{2x^2 + x - 1} dx$

e) $\int \frac{x^4 + 5x^2 + 2x + 6}{x(x^2 + 2)} dx$

- 7** a) Express $x^2 + 2x + 5$ in the form $(x + a)^2 + b$.

b) Find $\int \frac{dx}{x^2 + 4}$.

c) Hence find $\int \frac{dx}{x^2 + 2x + 5}$.

- 8** Integrate these rational functions.

a) $\frac{x}{x^2 + 6x + 25}$

b) $\frac{3x + 4}{x^2 + 8x + 20}$

Answers to AH Maths (MiA), pg. 113-4, Ex. 7.7

- 1 c** $3 \ln |x - 2| - \ln |x - 3| + c$
- d** $\frac{1}{3} \ln |3x + 1| + 4 \ln |x - 1| + c$
- e** $\frac{1}{3} \ln |x - 1| - \frac{1}{3} \ln |x + 2| + c$
- c** $8 \ln |x - 2| - 3 \ln |x - 1| + c$
- d** $2 \ln |x - 3| - \ln |x + 3| + c$
- c** $2 \ln |x| - \ln |x - 2| + \frac{1}{x-2} + c$
- d** $\ln |x - 1| - \ln |x - 2| - \frac{1}{x-2} + c$
- b** $\frac{1}{9} \ln |x - 1| - \frac{1}{9} \ln |x + 2| - \frac{2}{3(x+2)} + c$
- a** $\ln |x + 1| + \frac{1}{2} \ln |x^2 + 2| - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$
- b** $3 \ln |x| + \ln |x^2 + 1| + c$
- c** $2 \ln |x - 1| + \frac{1}{2} \ln |x^2 + 3| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + c$
- d** $\frac{1}{5} \ln |x| - \frac{1}{10} \ln |x^2 + 5| + c$
- e** $\frac{2}{7} \ln |x - 2| - \frac{1}{7} \ln |x^2 + 3| + \frac{3}{7\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + c$
- c** $x + 1 + \frac{-3x + 3}{(2x - 1)(x + 1)}, \frac{1}{2}x^2 + x + \frac{1}{2}$
 $\ln |2x - 1| - 2 \ln |x + 1| + c$
- e** $x + \frac{3x^2 + 2x + 6}{x(x^2 + 2)}, \frac{1}{2}x^2 + 3 \ln |x| + \sqrt{2}$
 $\tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$
- a** $(x + 1)^2 + 4$ **b** $\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$
- c** $\frac{1}{2} \tan^{-1} \left(\frac{x + 1}{2} \right) + c$
- a** $\frac{1}{2} \ln |x^2 + 6x + 25| - \frac{3}{4} \tan^{-1} \left(\frac{x + 3}{4} \right) + c$
- b** $\frac{3}{2} \ln |x^2 + 8x + 20| - 4 \tan^{-1} \left(\frac{x + 4}{2} \right) + c$