Differential Calculus - Lesson 3

Gradients of Tangent Lines

LI
  • Find gradients of tangent lines to curves.
  • Find a missing coordinate given the gradient.

SC
  • Differentiation.
The rate of change of \( y = f(x) \) at \( x = a \) (sometimes called the gradient of the curve at \( x = a \)) is equal to the gradient of the tangent line at \( x = a \):

\[
m = \left( \frac{dy}{dx} \right)_{x = a}
\]

Common notations for the gradient of the curve \( y = f(x) \) at \( x = a \) are:

- \( f'(a) \), \( y'(a) \) (Lagrange Form)
- \( \left( \frac{dy}{dx} \right)_{x = a} \) (Leibniz Form)
Useful Things to Remember
(especially for non-calc.)

\[ x^{1/2} = \sqrt{x} \]
\[ x^{3/2} = x \sqrt{x} \]
\[ x^{5/2} = x^2 \sqrt{x} \]
Example 1

Find the gradient of the tangent to the curve \( y = x^2 - 6x + 8 \) at the point \((2, 8)\).

\[
y(x) = x^2 - 6x + 8
\]

\[
\therefore \ y'(x) = 2x - 6
\]

\[
\therefore \ y'(2) = 2(2) - 6
\]

\[
\Rightarrow \ y'(2) = -2
\]
Example 2

A curve has equation \( y = 10 \sqrt{x} \).

Find the rate of change of \( y \) when \( x = 16 \).

\[
y(x) = 10 \sqrt{x}
\]

\[
y(x) = 10x^{1/2}
\]

\[
\therefore y'(x) = 5x^{-1/2}
\]

\[
\Rightarrow y'(x) = \frac{5}{x^{1/2}}
\]

\[
\Rightarrow y'(x) = \frac{5}{\sqrt{x}}
\]

\[
\therefore y'(16) = \frac{5}{\sqrt{16}}
\]

\[
\Rightarrow y'(16) = \frac{5}{4}
\]
Example 3

Find the gradient of the curve \( y = \frac{4}{\sqrt{x}} \)

at \( x = 4 \).

\[
y(x) = \frac{4}{\sqrt{x}}
\]

\[
y(x) = 4x^{-1/2}
\]

\[
\therefore \quad y'(x) = -2x^{-3/2}
\]

\[
\Rightarrow \quad y'(x) = -\frac{2}{x^{3/2}}
\]

\[
\Rightarrow \quad y'(x) = -\frac{2}{x\sqrt{x}}
\]

\[
\therefore \quad y'(4) = -\frac{2}{4\sqrt{4}}
\]

\[
\Rightarrow \quad y'(4) = -\frac{2}{8}
\]

\[
\Rightarrow \quad y'(4) = -\frac{1}{4}
\]
Example 4

A curve has equation \( y = 3x^2 - 12x + 6 \).

Find the \( x \)-coordinate of the point at which the tangent to the curve has gradient 12.

\[
\begin{align*}
y(x) &= 3x^2 - 12x + 6 \\
\therefore y'(x) &= 6x - 12
\end{align*}
\]

Gradient = 12 means \( y'(x) = 12 \). So,

\[
12 = 6x - 12
\]

\[
\Rightarrow 6x = 24
\]

\[
\Rightarrow x = 4
\]
CfE Higher Maths

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Questions

1 For each of the following, find the gradient of the curve at the given point.

a

\[ y = x^2 - 2x + 4 \]

\[ P (3, 7) \]

b

\[ y = x^2 - 4x - 5 \]

\[ Q (1, -8) \]

c

\[ y = -2 + 2x - x^2 \]

\[ R (0, -2) \]

d

\[ y = 6 - x - x^2 \]

\[ R (-2, 4) \]

2 a Find the gradient of the tangent to the curve \( y = x^2 + 4x + 2 \) at the point where \( x = 3 \).

b A curve has equation \( y = 5x^2 - 15x \). Find the gradient of the curve at the point where \( x = 2 \).

c Given \( f(x) = x^3 - 4x^2 + 5x + 3 \), find the rate of change of \( f \) when \( x = 1 \).

d Find the gradient of the curve \( y = (x + 2)(x + 5) \) at the point where \( x = -3 \).

e Given \( g(x) = 6x - x^3 \), find the value of \( g'(-2) \).

f A curve has equation \( y = 4x(x^2 - 2) \). Find \( \frac{dy}{dx} \) when \( x = -1 \).
3 A curve has equation \( y = \frac{2}{x} \) where \( x \neq 0 \). Find the gradient of the curve when
\[
\begin{align*}
\text{a} & \quad x = 1 \\
\text{b} & \quad x = -3 \\
\text{c} & \quad x = \frac{1}{2}
\end{align*}
\]
4 On a suitable domain, the function \( f \) is defined by \( f(x) = 3\sqrt{x} \)
\[
\begin{align*}
\text{a} & \quad \text{Find the gradient of the tangent to the curve } y = f(x) \text{ at the point where } x = 4. \\
\text{b} & \quad \text{Find the rate of change of } f \text{ when } x = 9. \\
\text{c} & \quad \text{Evaluate } f'(\frac{1}{16}).
\end{align*}
\]
5 The diagram shows part of the graph of the cubic function with equation \( f(x) = x(x^2 - 4) \). A tangent to the graph is drawn at \( P \).
Find the gradient of this tangent.
6 A curve has equation \( y = \frac{5}{4x^2} \) where \( x \neq 0 \).
Find the gradient of the curve at the point where \( x = -10 \).
7 \[
\begin{align*}
\text{a} & \quad \text{Find the } x-\text{coordinate of the point where the tangent to the curve } y = x^2 + 8x - 3 \text{ has gradient } 2.
\text{b} & \quad \text{The function } f \text{ is defined by } f(x) = 5 - 4x - x^2. \text{ Determine the value of } p, \text{ given that } f'(p) = 2.
\end{align*}
\]
8 Find the coordinates of the point where the tangent to the curve \( y = 3x^2 - 4x + 1 \) has gradient \(-10\).
9 Find the \( x-\)coordinate of the point where the tangent to the curve \( y = x^4 + 20x \) has gradient \(-12\).
10 \[
\begin{align*}
\text{a} & \quad \text{Determine the } x-\text{coordinates of the points where the tangent to the curve } y = \frac{1}{3}x^3 - 3x^2 + 12x + 2 \text{ has gradient } 4.
\text{b} & \quad \text{Determine the } x-\text{coordinates of the points where the tangent to the curve } y = x^3 + 2x^2 - 7x + 1 \text{ has gradient } -3.
\end{align*}
\]
18 Find the range of values of \( x \) for which the gradient of the curve \( y = x^3 + x^2 - 5x + 2 \) is greater than 3.
Answers

1  a  4
   b  -2
   c  2
   d  3
2  a  10
   b  5
   c  0
   d  1
   e  -6
   f  4
3  a  -2
   b  -\frac{2}{9}
   c  -8
4  a  \frac{3}{4}
   b  \frac{1}{2}
   c  6
5  -1
6  \frac{1}{400}
7  a  -3
   b  -3
8  (-1,8)
9  -2
10 a  x = 2
    x = 4
    x = -2
    x = \frac{2}{3}
18 x < -2 \text{ and } x > \frac{4}{3}