Polynomials - Lesson 3

The Factor Theorem

LI

- Factorise polynomials using the Factor Theorem.

SC

- Obtaining a zero remainder in synthetic division.

$$f(x) = Q(x)(x - h) + R$$

Factor Theorem

(i) $f(h) = 0$ (i.e. 0 remainder) $\Rightarrow$ $(x - h)$ is a factor of $f(x)$.

(ii) $(x - h)$ is a factor of $f(x) \Rightarrow f(h) = 0$ (i.e. 0 remainder).

Proof :

$$f(x) = Q(x)(x - h) + R$$

(i) $f(h) = 0$ means $R = 0$, so,

$$f(x) = Q(x)(x - h) + 0$$

$$\Rightarrow \quad f(x) = Q(x)(x - h)$$

i.e., $(x - h)$ is a factor of $f(x)$

(ii) $(x - h)$ is a factor of $f(x)$ means,

$$f(x) = Q(x)(x - h)$$

$$\therefore \quad f(h) = Q(h)(h - h)$$

$$\Rightarrow \quad f(h) = 0$$

Example 1

Show that $(x - 4)$ is a factor of the quartic $f(x) = 2x^4 - 9x^3 + 5x^2 - 3x - 4$.

	x^4	x^3	x^2	x^1	x^0
4	2	− 9	5	− 3	− 4
		8	− 4	4	4
	2	− 1	1	1	0

As $R = 0$, by the Factor Theorem, $(x - 4)$ is a factor of $f(x)$.

How to Factorise a Polynomial

Strategy

- Look for factors of constant term (+ ve and − ve).
- If remainder is not 0 with a chosen factor, try another factor (start with small factors).
- Keep going until remainder is 0.
- Factorise fully the quotient.

Example 2

Factorise fully $f(x) = 2x^3 + 5x^2 - 28x - 15$.

Factors of 15 : $\pm 1, \pm 3, \pm 5, \pm 15$.

Try 1 :

	x^3	x^2	x^1	x^0	
1	2	5	− 28	− 15	
		2	7	− 21	
	2	7	− 21	**− 36**	×

Try − 1 :

	x^3	x^2	x^1	x^0	
− 1	2	5	− 28	− 15	
		− 2	− 3	31	
	2	3	− 31	**16**	×

Try 3 :

	x^3	x^2	x^1	x^0
3	2	5	− 28	− 15
		6	33	15
	2	11	5	**0**

As $R = 0$, by the Factor Theorem, $(x - 3)$ is a factor of $f(x)$. So,

$$f(x) = (x - 3)(2x^2 + 11x + 5)$$

Now factorise the quadratic :

$$2x^2 + 11x + 5 = (2x + 1)(x + 5)$$

Hence,

$$\boxed{f(x) = (x - 3)(2x + 1)(x + 5)}$$

Example 3

Determine the values of b and d if $(x - 1)$ and $(x + 3)$ are both factors of the cubic $f(x) = 2x^3 + bx^2 - 4x + d$ and hence fully factorise $f(x)$.

	x^3	x^2	x^1	x^0
1	2	b	-4	d
		2	$b + 2$	$b - 2$
	2	$b + 2$	$b - 2$	$b + d - 2$

As $R = 0$,

$$b + d - 2 = 0$$

$$\Rightarrow \quad b + d = 2 \qquad (1)$$

	x^3	x^2	x^1	x^0
-3	2	b	-4	d
		-6	$18 - 3b$	$9b - 42$
	2	$b - 6$	$14 - 3b$	$9b + d - 42$

As $R = 0$,

$$9b + d - 42 = 0$$

$$\Rightarrow \quad 9b + d = 42 \qquad (2)$$

$$b + d = 2 \qquad (1)$$

$$9b + d = 42 \qquad (2)$$

$(2) - (1)$ gives,

$$8b = 40 \Rightarrow \boxed{b = 5}$$

$$b + d = 2 \qquad (1)$$

$$5 + d = 2 \Rightarrow \boxed{d = -3}$$

Hence, using the first synthetic division procedure for the coefficients of the quotient (can use the other one too),

$$f(x) = (x - 1)(2x^2 + 7x + 3)$$

$$\Rightarrow \boxed{f(x) = (x - 1)(2x + 1)(x + 3)}$$

CfE Higher Maths

pg. 138 Ex. 7A Q 7 - 9

pg. 144 Ex. 7B Q 2, 5, 8

pg. 145 Ex. 7C Q 3 c

pg. 149 - 50 Ex. 7D Q 3 d, 4 h, 5

pg. 154 Ex. 7E Q 2, 3, 6

Questions - Ex. 7A

7 Determine whether or not $x + 2$ is a factor of $f(x) = 2x^3 + 3x^2 - 2x - 1$

8 Determine whether or not $x - 1$ is a factor of $f(x) = 2x^5 + 3x^2 + 2x + 1$

9 Determine whether or not $x - \frac{1}{2}$ is a factor of $f(x) = 4x^3 + x^2 - x - \frac{1}{4}$

Answers

7 no

8 no

9 yes

Questions - Ex. 7B

2 Show that $x + 4$ is a factor of $f(x) = x^3 + 4x^2 - x - 4$ and factorise fully.

5 Show that $x + 3$ is a factor of $f(x) = x^3 - 13x - 12$ and factorise fully.

8 Prove that $x + 5$ is a factor of $f(x) = x^4 + x^3 - 16x^2 + 20x$ and express $f(x)$ in factorised form.

Answers

2 $(x + 1)(x - 1)(x + 4)$

5 $(x + 3)(x + 1)(x - 4)$

8 $(x + 5)(x - 2)(x - 2)x$

Questions - Ex. 7C

3 Express $f(x)$ in factorised form.

c $f(x) = x^4 - 3x^3 - 6x^2 + 8x$

Answers

3 c $(x - 4)(x - 1)(x + 2)x$

Questions - Ex. 7D

3 Fully factorise each quartic:

d $f(x) = x^4 - 5x^3 + x^2 + 21x - 18$

4 Fully factorise each polynomial:

h $f(x) = 4x^4 + 4x^3 - 9x^2 - x + 2$

5 A company carves candles from cuboids of wax with whole number dimensions. The volume of a cuboid, in cm^3, is given by the function $V(x) = x^3 - 10x^2 + 13x + 24$ and the height of the cuboid is $x + 1cm$. Determine expressions for the other two dimensions of the cuboid.

Answers

3 d $(x-3)(x-3)(x-1)(x+2)$

4 h $(2x-1)(x-1)(2x+1)(x+2)$

5 $(x-8)(x+1)(x-3)$

Questions - Ex. 7E

2 Find q if $x - 2$ is a factor of $f(x) = x^3 - 9x^2 + 24x - q$

3 $f(x) = 2kx^3 + kx^2 - 2x - 1$ has $x - 1$ as a factor. Determine the value of k and fully factorise $f(x)$.

6 $x + 1$ and $x - 2$ are factors of $f(x) = ax^3 - 3x^2 - 3x + b$. Determine the values of a and b.

Answers

2 $q = 20$

3 $k = 1$

$(x - 1)(x + 1)(2x + 1)$

6 $a = 2$

$b = 2$