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Vectors, Lines and Planes - Lesson 3

Equations of Lines and Their Intersection

LI

- Obtain an equation for a line in Vector, Parametric and Symmetric Form.
- Work out points and angles where lines intersect.

SC

- Scalar product.
- Geometric intuition.

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A line is an infinite 1D space

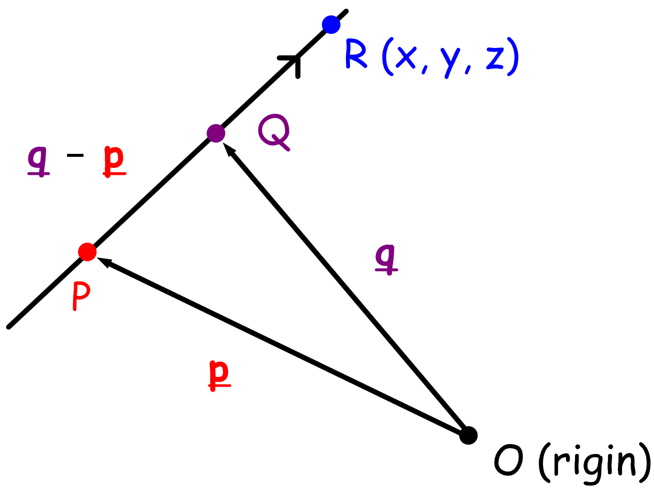
An equation of a line can be found in 2 ways :

- 1 point on the line and a direction vector for the line.
- 2 points on the line.

A vector is parallel to a line if it lies on the line

Equations of a Line

$$\underline{d} = \underline{q} - \underline{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a, b, c)^T$$



The vector equation of a line (with parameter t) is,

$$\underline{r} = \underline{p} + t\,\underline{d}$$

'To get to $\underline{r}$, go to $\underline{p}$, then go a little bit along $\underline{d}$.'

If P has coordinates (x_0, y_0, z_0) and the direction vector $\underline{d} = (a, b, c)^T$, the parametric equations of the line are :

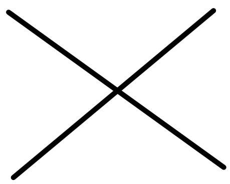
$$x = x_0 + t a$$
$$y = y_0 + t b$$
$$z = z_0 + t c$$

Assuming that none of a, b or c equals 0, these equations can be written in symmetric form :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (= t \in \mathbb{R})$$

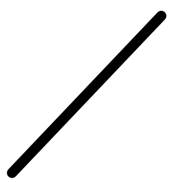
Intersections of Lines

Unique Intersection Point



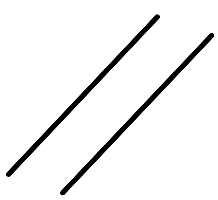
Unique solutions for line parameters t and u

Infinitely Many Intersection Points



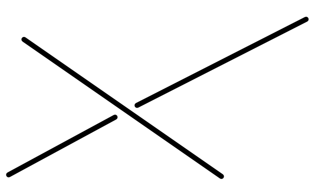
Infinitely many solutions for t and u

No Intersection Points (Parallel)



No solutions for t and u
(direction vectors parallel)

No Intersection Points (Skew)



(think 3D)

No solutions for t and u
(direction vectors not parallel)

The angle between two lines is the angle between their direction vectors

Example 1 (Equation given point and direction)

Find the vector, parametric and symmetric equations of the line through $F\,(4, 11, 17)$ and which is parallel to $3\,\underline{i} + 5\,\underline{j} - 8\,\underline{k}$.

$$\underline{r} = \underline{p} + t\,\underline{d}$$

Vector form : $(x, y, z)^T = (4, 11, 17)^T + t\,(3, 5, -8)^T$

Parametric form :

$$x = 4 + 3t$$
$$y = 11 + 5t$$
$$z = 17 - 8t$$

Symmetric form : $\dfrac{x - 4}{3} = \dfrac{y - 11}{5} = \dfrac{z - 17}{-8} \quad (= t \in \mathbb{R})$

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Example 2 (Equation given 2 points)

Find a vector form of the line passing through $A\ (1, 4, 6)$ and $B\ (3, -3, -3)$.

A direction vector for the line is $\overrightarrow{AB} = (2, -7, -9)^T$. Hence,

$$\boxed{(x, y, z)^T = (1, 4, 6)^T + t\,(2, -7, -9)^T}$$

Example 3

Find the value of k so that the lines,

$$L_1: \quad \frac{x + 27}{6} = \frac{y - 4}{5} = \frac{z}{-3}$$

$$L_2: \quad \frac{x - 12}{1} = \frac{y - 39}{3} = \frac{z - 2}{k}$$

are perpendicular.

For the lines to be perpendicular, their direction vectors must have zero scalar product. Hence,

$$(6)(1) + (5)(3) + (-3)(k) = 0$$

$$\Rightarrow \quad 21 - 3k = 0$$

$$\Rightarrow \quad \boxed{k = 7}$$

Example 4

Find the acute angle between the lines,

$$L_1: \quad \frac{x+27}{2} = \frac{y-4}{1} = \frac{z}{-3}$$

$$L_2: \quad \frac{x-12}{1} = \frac{y-39}{3} = \frac{z-2}{6}$$

Using the scalar product formula gives,

$$\cos\theta = -\frac{13}{\sqrt{14}\sqrt{46}}$$

$$\therefore \quad \underline{RAA = 59.2^\circ}$$

As the acute angle is required,

$$\boxed{\theta = 59.2^\circ}$$

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Example 5

Show that the lines,

$$L_1 : \quad \frac{x - 12}{5} = \frac{y + 3}{-2} = \frac{z - 5}{4} \quad (= t \in \mathbb{R})$$

$$L_2 : \quad x - 5 = -(y + 2) = z \quad (= u \in \mathbb{R})$$

intersect and find the point(s) of intersection.

The lines can be written in parametric form as,

$$\begin{aligned} x &= 5t + 12 \qquad & x &= u + 5 \\ y &= -2t - 3 \qquad & y &= -u - 2 \\ z &= 4t + 5 \qquad & z &= u \end{aligned}$$

For the lines to intersect, there must be at least one common (x, y, z) coordinate; this implies that,

$$\begin{aligned} u + 5 &= 5t + 12 \\ -u - 2 &= -2t - 3 \\ u &= 4t + 5 \end{aligned}$$

and hence,

$$\begin{aligned} 5t - u &= -7 \\ 2t - u &= -1 \\ 4t - u &= -5 \end{aligned} \qquad ★$$

Solving these gives $t = -2$ and $u = -3$.

$\boxed{\text{As the equations ★ are consistent, the lines intersect}}$

As there is a unique set of solutions for t and u, there will be only 1 intersection point. Substituting either t or u into the respective parametric equations gives the point of intersection.

$\boxed{\text{Intersection point : } (2, 1, -3)}$

Example 6

Show that the lines,

$$L_1: \quad \frac{x-12}{5} = \frac{y+3}{-2} = \frac{z-5}{4} \quad (= t \in \mathbb{R})$$

$$L_2: \quad \frac{x-12}{10} = \frac{y+3}{-4} = \frac{z-5}{8} \quad (= u \in \mathbb{R})$$

intersect in infinitely many points.

The parametric equations give (check !) $t = 2u$ (thrice). Hence, there are infinitely many solutions for t and u. Notice that the lines both pass through the point $(12, -3, 5)$ and have parallel direction vectors (size doesn't matter when comparing line direction vectors). There is really only 1 line !

$$\boxed{\text{Intersection points}: \ (5t + 12, -2t - 3, 4t + 5) \ (t \in \mathbb{R})}$$

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Example 7

Show that the lines,

$$L_1: \quad \frac{x-12}{5} = \frac{y+3}{-2} = \frac{z-5}{4} \quad (= t \in \mathbb{R})$$

$$L_2: \quad \frac{x-3}{10} = \frac{y+7}{-4} = \frac{z-1}{8} \quad (= u \in \mathbb{R})$$

are parallel with no intersection points.

The lines are clearly parallel, as their direction vectors are $(5, -2, 4)^T$ and $(10, -4, 8)^T = 2\,(5, -2, 4)^T$. The parametric equations are,

$$\begin{aligned} x &= 5t + 12 \\ y &= -2t - 3 \\ z &= 4t + 5 \end{aligned} \qquad\qquad \begin{aligned} x &= 10u + 3 \\ y &= -4u - 7 \\ z &= 8u + 1 \end{aligned}$$

These simplify to give,

$$\begin{aligned} 5t - 10u &= -9 \\ t - 2u &= 2 \qquad ★ \\ t - 2u &= -1 \end{aligned}$$

The last 2 equations are clearly inconsistent; hence, no common intersection point.

As the equations ★ are inconsistent, and the direction vectors are parallel, the lines are parallel with no intersection point

Example 8

Show that the lines,

$$L_1: \quad \frac{x - 12}{2} = \frac{y + 3}{-2} = \frac{z - 5}{4} \quad (= t \in \mathbb{R})$$

$$L_2: \quad \frac{x - 3}{1} = \frac{y + 7}{-3} = \frac{z - 1}{2} \quad (= u \in \mathbb{R})$$

are skew.

To show that the lines are skew, it must be shown that the direction vectors are not parallel and there is no common intersection point. The direction vectors are $(2, -2, 4)^T$ and $(1, -3, 2)^T \neq k\,(2, -2, 4)^T$; hence, the lines are not parallel. The parametric equations are,

$$\begin{aligned} x &= 2t + 12 \\ y &= -2t - 3 \\ z &= 4t + 5 \end{aligned} \qquad\qquad \begin{aligned} x &= u + 3 \\ y &= -3u - 7 \\ z &= 2u + 1 \end{aligned}$$

These simplify to give,

$$\begin{aligned} 2t - u &= -9 \\ 2t - 3u &= 4 \\ 2t - u &= -2 \end{aligned} \qquad \bigstar$$

The first and last equations are clearly inconsistent; hence, no common intersection point.

$\boxed{\text{As the equations } \bigstar \text{ are inconsistent, and the direction vectors are not parallel, the lines are skew}}$

AH Maths - MiA (2nd Edn.)

- pg. 298 Ex. 15.8
  Q 1, 2, 3 a - c, 4, 5.
- pg. 298-9 Ex. 15.9
  Q 2.
- pg. 302 Ex. 15.11
  Q 1 a (i) - (iii), b (i) - (iii).

Ex. 15.8

1 Write the system of equations which represents the line which passes through the point

a $(1, -2, 3)$ with direction vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ b $(-1, 2, -2)$ with direction vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$

c $(4, 2, -1)$ with direction vector $3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

2 Write the equations of the line which passes through the point

a $(5, -3, -1)$ and is parallel to $2\mathbf{i} + \mathbf{k}$ b $(0, 0, 0)$ and is parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

c $(0, 0, 0)$ and is parallel to $\mathbf{k}$.

3 Write the equations of the line which passes through the points

a $(0, 1, 3)$ and $(-1, 2, -4)$ b $(5, -1, 0)$ and $(6, 2, -7)$ c $(3, 11, -2)$ and $(6, -1, 0)$

4 Show that the point $A(2, 3, -4)$ lies on both the lines with equations

$\frac{x+2}{2} = \frac{y-1}{1} = \frac{z-2}{-3}$ and $\frac{x+4}{2} = \frac{y+3}{2} = \frac{z+7}{1}$, and show that the angle between these lines is $\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$.

5 Show that the lines with equations

$\frac{x+5}{2} = \frac{y+7}{-1} = \frac{z-1}{4}$ and $\frac{x-3}{3} = \frac{y+6}{-2} = \frac{z+7}{-2}$ are perpendicular.

Ex. 15.9

2 Find a vector equation for AB where

a $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

b A is the point $(-1, 2, 0)$ and B is $(2, 3, -1)$

c $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the line AB is parallel to $3\mathbf{i} - \mathbf{j} - \mathbf{k}$

d B is the point $(2, 1, -1)$ and AB is parallel to the join of $(1, 0, -1)$ and $(3, 1, 2)$.

Ex. 15.11

1 For each of these pairs of lines

i prove that the lines intersect

ii find the point of intersection

iii find the acute angle between the lines (to 3 sf)

a $\dfrac{x+13}{4} = \dfrac{y-7}{-3} = \dfrac{z+1}{2}$ and $\dfrac{x+13}{-3} = \dfrac{y-6}{2} = \dfrac{z-21}{4}$

b $\dfrac{x-19}{4} = \dfrac{y+6}{-3} = \dfrac{z-13}{5}$ and $x - 8 = 2 - y = \dfrac{z}{2}$

Answers to AH Maths (MiA), pg. 298, Ex. 15.8

1 **a** $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z-3}{-1}$ **b** $\frac{x+1}{1}=\frac{y-2}{-1}=\frac{z+2}{1}$

c $\frac{x-4}{3}=\frac{y-2}{1}=\frac{z+1}{3}$

2 **a** $x=2t+5, y=-3, z=t-1$

b $\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$ **c** $x=0, y=0, z=t$

3 **a** $\frac{x}{1}=\frac{y-1}{-1}=\frac{z-3}{7}$ **b** $\frac{x-5}{1}=\frac{y+1}{3}=\frac{z}{-7}$

c $\frac{x-3}{3}=\frac{y-11}{-12}=\frac{z+2}{2}$

4 Verification

5 Verification

Answers to AH Maths (MiA), pg. 298-9, Ex. 15.9

2 **a** $\mathbf{r}=(1+t)\mathbf{i}+(1-2t)\mathbf{j}-\mathbf{k}$

b $\mathbf{r}=(3t-1)\mathbf{i}+(2+t)\mathbf{j}-t\mathbf{k}$

c $\mathbf{r}=(2+3t)\mathbf{i}+(2-t)\mathbf{j}+(1-t)\mathbf{k}$

d $\mathbf{r}=(2t+2)\mathbf{i}+(t+1)\mathbf{j}+(3t-1)\mathbf{k}$

Answers to AH Maths (MiA), pg. 302, Ex. 15.11

1 **a** **i** Proof **ii** $(-1, -2, 5)$ **iii** $69{\cdot}8°$

b **i** Proof **ii** $(7, 3, -2)$ **iii** $11{\cdot}0°$