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Unit 1 : Differential Calculus - Lesson 3

Derivatives of Reciprocal Trigonometric Functions and Tan x

LI

- Know the 3 Reciprocal Trigonometric Functions and their derivatives.
- Know the derivative of tan x.
- Prove the derivatives of the above 4 functions.
- Learn 2 new trigonometric identities.

SC

- Chain and Quotient Rules.

Reciprocal Trigonometric Functions and Their Derivatives

$$\sec x \stackrel{def}{=} \frac{1}{\cos x}$$

$$\operatorname{cosec} x \stackrel{def}{=} \frac{1}{\sin x}$$

$$\cot x \stackrel{def}{=} \frac{\cos x}{\sin x}$$

- $$\frac{d}{dx} \sec x = \sec x \tan x$$
- $$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$$
- $$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

'sec'	is short for	'secant'
'cosec'	''	'cosecant'
'cot'	''	'cotangent'

Derivative of Tan x

- $$\frac{d}{dx} \tan x = \sec^2 x$$

General Form of Derivatives - Chain Rule

$$\frac{d}{dx}\sec f(x) = \sec f(x) \,.\, \tan f(x) \,.\, f'(x)$$

$$\frac{d}{dx}\operatorname{cosec} f(x) = -\operatorname{cosec} f(x) \,.\, \cot f(x) \,.\, f'(x)$$

$$\frac{d}{dx}\cot f(x) = -\operatorname{cosec}^2 f(x) \,.\, f'(x)$$

$$\frac{d}{dx}\tan f(x) = \sec^2 f(x) \,.\, f'(x)$$

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2 New Trigonometric Identities

$$\underbrace{\sin^2 x + \cos^2 x = 1}$$

$\div \cos^2 x$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

$\div \sin^2 x$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\boxed{1 + \cot^2 x = \operatorname{cosec}^2 x}$$

Example 1

Show that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

Let $y = \operatorname{cosec} x = \frac{1}{\sin x}$.

$$\boxed{\begin{array}{ll} f(x) = 1 , & g(x) = \sin x \\ f'(x) = 0 , & g'(x) = \cos x \end{array}}$$

$$y' = \frac{f'g - fg'}{g^2}$$

$$\therefore \quad y' = \frac{0 . \sin x - 1 . \cos x}{(\sin x)^2}$$

$$\Rightarrow \quad y' = \frac{-\cos x}{(\sin x)(\sin x)}$$

$$\Rightarrow \quad y' = -\frac{1}{\sin x} . \frac{\cos x}{\sin x}$$

$$\Rightarrow \quad \boxed{y' = -\operatorname{cosec} x \cot x}$$

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Example 2

If $f(x) = \sec 7x$, find $f'(x)$.

$$f(x) = \sec 7x$$

$$\therefore \quad f'(x) = \sec 7x \,.\, \tan 7x \,.\, \frac{d}{dx}(7x)$$

$$\Rightarrow \quad \boxed{f'(x) = 7 \sec 7x \,.\, \tan 7x}$$

Example 3

If $g(p) = \tan(9 - 5p)$, find $g'(p)$.

$$g(p) = \tan(9 - 5p)$$

$$\therefore \quad g'(p) = \sec^2(9 - 5p) \cdot \frac{d}{dp}(9 - 5p)$$

$$\Rightarrow \quad \boxed{g'(p) = -5\sec^2(9 - 5p)}$$

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Example 4

If $y = \cos(\cot x)$, find y'.

$$y = \cos(\cot x)$$

$$\therefore \quad y' = -\sin(\cot x) \cdot \frac{d}{dx}(\cot x)$$

$$\Rightarrow \quad y' = -\sin(\cot x) \cdot (-\operatorname{cosec}^2 x)$$

$$\Rightarrow \quad \boxed{y' = \operatorname{cosec}^2 x \sin(\cot x)}$$

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Example 5

If $y = \cos x \cot x$, find y'.

$$y = \cos x \cot x$$

$$\therefore \quad y' = -\sin x \,.\, \cot x + \cos x \,.\, (-\operatorname{cosec}^2 x)$$

$$\Rightarrow \quad \boxed{\begin{array}{l} y' = -\sin x \cot x - \cos x \operatorname{cosec}^2 x \\ \left(y' = -\cos x - \cot x \operatorname{cosec} x \right) \end{array}}$$

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AH Maths - MiA (2^{nd} Edn.)

- pg. 55-6 Ex. 4.8 All Q.

Ex. 4.8

1 Differentiate and simplify **a** $\operatorname{cosec} x$ **b** $\cot x$

2 Find the derivative of each of these.

a $\sec 2x$ **b** $\tan 2x$ **c** $\operatorname{cosec} 2x$ **d** $\operatorname{cosec}(2x + 3)$

e $\sec(4 - 3x^2)$ **f** $\cot 5x$ **g** $\cot(x^2)$ **h** $\tan(1 - 17x)$

3 Calculate $\frac{dy}{dx}$ in each case.

a $y = \sec x \tan x$ **b** $y = \cot(\tan x)$ **c** $y = \operatorname{cosec}(\sin x)$

d $y = \operatorname{cosec}^2 3x$ **e** $y = \sec^2 x$ **f** $y = \tan^2 4x$

g $y = \sqrt{\sec x}$ **h** $y = \dfrac{1}{\sqrt{1 + \operatorname{cosec} x}}$

4 Find $f'(x)$ when

a $f(x) = \dfrac{x^2 + x}{1 + \cot x}$ **b** $f(x) = \dfrac{\cot x + \sec x}{\cot x - \sec x}$ **c** $f(x) = \dfrac{\sec x + \cot x}{x^2 + 2x + 1}$

5 Given that $f(x) = \sin^2 x \tan x$, show that $f'\left(\frac{\pi}{4}\right) = 2$.

6 **a** If $f(x) = \sin x \sec x$, show that $f'\left(\frac{\pi}{3}\right) = 4$.

b How might $f(x)$ have been simplified to make the problem easier?

Answers to AH Maths (MiA), pg. 55-6, Ex. 4.8

1 **a** $-\cos x \operatorname{cosec}^2 x$ **b** $-\operatorname{cosec}^2 x$

2 **a** $2 \sin 2x \sec^2 2x$ **b** $2 \sec^2 2x$

c $-2 \cos 2x \operatorname{cosec}^2 2x$

d $-2 \cos(2x + 3) \operatorname{cosec}^2 (2x + 3)$

e $-6x \sin(4 - 3x^2) \sec^2 (4 - 3x^2)$

f $-5 \operatorname{cosec}^2 5x$

g $-2x \operatorname{cosec}^2 (x^2)$

h $-17 \sec^2 (1 - 17x)$

3 **a** $\sec x \tan^2 x + \sec^3 x$

b $-\operatorname{cosec}^2 (\tan x) \sec^2 x$

c $-\operatorname{cosec} (\sin x) \cot (\sin x) \cos x$

d $-6 \operatorname{cosec}^2 3x \cot 3x$

e $2 \sec^2 x \tan x$

f $8 \tan 4x \sec^2 4x$

g $\frac{1}{2} \sqrt{\sec x} \,.\, \tan x$

h $\frac{1}{2}(1 + \operatorname{cosec} x)^{-\frac{3}{2}} \operatorname{cosec} x \cot x$

4 **a** $\dfrac{(1 + \cot x)(2x + 1) + (x^2 + x) \operatorname{cosec}^2 x}{(1 + \cot x)^2}$

b $\dfrac{2 \sec x (1 + \operatorname{cosec}^2 x)}{(\cot x - \sec x)^2}$

c $\dfrac{(x + 1)(\sec x \tan x - \operatorname{cosec}^2 x) - 2(\sec x + \cot x)}{(x + 1)^3}$

5 $f'(x) = 2 \sin x \cos x \tan x + \sin^2 x \sec^2 x$, hence result.

6 **a** $f'(x) = \cos x \sec x + \sin x \sec x \tan x$
$= 1 + \sin x \sec x \tan x$ hence result.

b $\sin x \sec x = \tan x$