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Geometry of Complex Numbers - Lesson 3

De Moivre's Theorem and Powers

LI

- Prove De Moivre's Theorem using Mathematical Induction.
- Take powers of a complex number.
- Prove trigonometric identities using the Binomial Theorem and de Moivre's Theorem.

SC

- Polar form.
- Mathematical induction.
- Binomial Theorem.

De Moivre's Theorem states that,

$$z = r(\cos \theta + i \sin \theta) \Rightarrow z^n = r^n (\cos n\theta + i \sin n\theta) (\forall n \in \mathbb{R})$$

Short form: $z = r \operatorname{cis} \theta \Rightarrow z^n = r^n (\operatorname{cis} n\theta) (\forall n \in \mathbb{R})$

Example 1

Prove de Moivre's Theorem for all natural numbers n .

We use mathematical induction.

$$P(n) : [r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta)$$

Base Case

$$LHS = [r(\cos \theta + i \sin \theta)]^1 = r(\cos \theta + i \sin \theta)$$

$$RHS = r^1 (\cos(1\theta) + i \sin(1\theta)) = r(\cos \theta + i \sin \theta)$$

As LHS = RHS, $P(1)$ is true

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$[r(\cos \theta + i \sin \theta)]^k = r^k (\cos k\theta + i \sin k\theta) \leftarrow \begin{matrix} \text{Inductive} \\ \text{Hypothesis} \end{matrix}$$

RTP statement

$$[r(\cos \theta + i \sin \theta)]^{k+1} = r^{k+1} (\cos(k+1)\theta + i \sin(k+1)\theta)$$

$$\begin{aligned} & [r(\cos \theta + i \sin \theta)]^{k+1} \\ &= [r(\cos \theta + i \sin \theta)]^k [r(\cos \theta + i \sin \theta)] \\ &= [r^k (\cos k\theta + i \sin k\theta)] [r(\cos \theta + i \sin \theta)] \\ &= r^{k+1} (\cos k\theta + i \sin k\theta) (\cos \theta + i \sin \theta) \\ &= r^{k+1} (\cos k\theta \cos \theta + i \cos k\theta \sin \theta + i \sin k\theta \cos \theta \\ &\quad + i^2 \sin k\theta \sin \theta) \\ &= r^{k+1} (\cos k\theta \cos \theta - \sin k\theta \sin \theta) \\ &\quad + i r^{k+1} (\sin k\theta \cos \theta + \cos k\theta \sin \theta) \\ &= r^{k+1} (\cos(k\theta + \theta)) + i r^{k+1} (\sin(k\theta + \theta)) \\ &= r^{k+1} (\cos(k+1)\theta) + i r^{k+1} (\sin(k+1)\theta) \\ &= r^{k+1} (\cos(k+1)\theta) + i \sin(k+1)\theta \end{aligned}$$

Hence, $P(k)$ true $\Rightarrow P(k+1)$ true

'P(1) true' and 'P(k) true $\Rightarrow P(k+1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

Example 2

If $z = 1 + i\sqrt{3}$, find z^{10} in Cartesian form.

Writing z in polar form gives (check !),

$$z = 2 \operatorname{cis}(\pi/3)$$

$$\therefore z^{10} = (2 \operatorname{cis}(\pi/3))^{10}$$

$$\Rightarrow z^{10} = 2^{10} \operatorname{cis}(10\pi/3) \quad \text{de Moivre's Theorem}$$

$$\Rightarrow z^{10} = 2^{10} \operatorname{cis}(-2\pi/3) \quad \begin{matrix} 10\pi/3 \text{ not in range} \\ \text{so subtract } 2\pi \text{'s to} \\ \text{get it into range} \end{matrix}$$

$$\Rightarrow z^{10} = 1024 (\cos(-2\pi/3) + i \sin(-2\pi/3))$$

$$\Rightarrow z^{10} = 1024 ((-1/2) + i (-\sqrt{3}/2))$$

$$\Rightarrow z^{10} = -512 - 512\sqrt{3}i$$

Example 3

Simplify fully,

$$\begin{aligned}
 & \frac{(\cos 42^\circ + i \sin 42^\circ)^2}{(\cos 7^\circ + i \sin 7^\circ)^3 (\cos 10^\circ + i \sin 10^\circ)^4} \\
 \\
 & \frac{(\operatorname{cis} 42^\circ)^2}{(\operatorname{cis} 7^\circ)^3 (\operatorname{cis} 10^\circ)^4} \\
 \\
 & = \frac{\operatorname{cis} 84^\circ}{(\operatorname{cis} 21^\circ) (\operatorname{cis} 40^\circ)} \\
 \\
 & = \frac{\operatorname{cis} 84^\circ}{\operatorname{cis} 61^\circ} \\
 \\
 & = \operatorname{cis} 23^\circ \\
 \\
 & = \boxed{\cos 23^\circ + i \sin 23^\circ}
 \end{aligned}$$

Example 4

By expanding $(\cos \theta + i \sin \theta)^5$ using the Binomial Theorem and by using de Moivre's Theorem, express $\cos 5\theta$ in terms of powers of $\cos \theta$ only.

First make the (unbelievably useful !) abbreviations,

$$C = \cos \theta, S = \sin \theta$$

De Moivre's Theorem gives,

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$$

The Binomial Theorem gives,

$$\begin{aligned} (\cos \theta + i \sin \theta)^5 &= C^5 + 5C^4(iS)^1 + 10C^3(iS)^2 \\ &\quad + 10C^2(iS)^3 + 5C^1(iS)^4 + (iS)^5 \end{aligned}$$

$$\Rightarrow (\cos \theta + i \sin \theta)^5 = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$$

$$\Rightarrow (\cos \theta + i \sin \theta)^5 = (C^5 - 10C^3S^2 + 5CS^4) + i(5C^4S - 10C^2S^3 + S^5)$$

Equating the two expressions for $(\cos \theta + i \sin \theta)^5$ gives,

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= (C^5 - 10C^3S^2 + 5CS^4) \\ &\quad + i(5C^4S - 10C^2S^3 + S^5) \end{aligned}$$

Equating real parts gives,

$$\cos 5\theta = C^5 - 10C^3S^2 + 5CS^4$$

$$\Rightarrow \cos 5\theta = C^5 - 10C^3(1 - C^2) + 5C(1 - C^2)^2$$

$$\Rightarrow \cos 5\theta = C^5 - 10C^3(1 - C^2) + 5C(1 - 2C^2 + C^4)$$

$$\Rightarrow \cos 5\theta = C^5 - 10C^3 + 10C^5 + 5C - 10C^3 + 5C^5$$

$$\Rightarrow \cos 5\theta = 16C^5 - 20C^3 + 5C$$

$$\Rightarrow \boxed{\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta}$$

AH Maths - MiA (2nd Edn.)

- pg. 218 - 9 Ex. 12.6
Q 1 c, 2, 4, 6, 7 a, 9.

Ex. 12.6

1 For each of these complex numbers,

- express it in polar form
- express each of the required powers in polar form, bringing the argument into the range $(-\pi, \pi]$
- finally express your answer in the form $a + ib$.

c Given $z = 1 + i$ find i z^3 ii z^6 iii z^{12}

2 Simplify these expressions,

a $[3(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})]^3$

b $[2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})]^8$

c $(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^2 (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})^2$

d $(\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7})^3 (\cos \frac{3\pi}{7} + i \sin \frac{3\pi}{7})^4$

4 The argument can be given in degrees.

The same laws apply and the range of the argument is $(-180^\circ, 180^\circ]$.

Simplify these expressions.

a $(\cos 10^\circ + i \sin 10^\circ)(\cos 30^\circ + i \sin 30^\circ)$

b $(\cos 50^\circ + i \sin 50^\circ)(\cos 145^\circ + i \sin 145^\circ)$

c $(\cos 25^\circ + i \sin 25^\circ)(\cos 20^\circ - i \sin 20^\circ)$

d $(\cos 150^\circ - i \sin 150^\circ)(\cos 40^\circ - i \sin 40^\circ)$

e $(\cos 30^\circ + i \sin 30^\circ) \div (\cos 10^\circ + i \sin 10^\circ)$

f $(\cos 4^\circ + i \sin 4^\circ) \div (\cos 10^\circ + i \sin 10^\circ)$

g $(\cos 20^\circ + i \sin 20^\circ)^3 (\cos 30^\circ + i \sin 30^\circ)^2$

h $(\cos 125^\circ - i \sin 125^\circ)^4 (\cos 15^\circ - i \sin 15^\circ)^3$

i $\frac{(\cos 40^\circ + i \sin 40^\circ)^3}{(\cos 10^\circ + i \sin 10^\circ)^2}$

j $\frac{(\cos 6^\circ + i \sin 6^\circ)^5}{(\cos 3^\circ + i \sin 3^\circ)^2}$

k $\frac{(\cos 25^\circ + i \sin 25^\circ)^4}{(\cos 7^\circ + i \sin 7^\circ)(\cos 3^\circ + i \sin 3^\circ)}$

l $\frac{(\cos 32^\circ + i \sin 32^\circ)^4}{(\cos 16^\circ + i \sin 16^\circ)^3 (\cos 4^\circ - i \sin 4^\circ)^2}$

6 a Expand $(\cos \theta + i \sin \theta)^3$

i using the binomial theorem ii using De Moivre's theorem.

b i By equating the real parts express $\cos 3\theta$ in terms of $\sin \theta$ and $\cos \theta$.

ii Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ to help you express $\cos 3\theta$ in terms of $\cos \theta$ only.

c Express $\sin 3\theta$ in terms of $\sin \theta$.

d Hence, express $\sin^3 \theta$ in terms of $\sin \theta$ and $\sin 3\theta$.

7 a By considering the expansion of $(\cos \theta + i \sin \theta)^4$ express

i $\cos 4\theta$ in terms of $\cos \theta$

ii $\sin 4\theta$ in terms of $\sin \theta$ and $\cos \theta$

iii $\cos^4 \theta$ in terms of $\cos \theta$ and $\cos 4\theta$.

9 $z_1 = \cos \frac{13\pi}{12} - i \sin \frac{13\pi}{12}, z_2 = \cos \frac{5\pi}{12} - i \sin \frac{5\pi}{12}, z_3 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4},$

$z_4 = \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}, z_5 = \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12}$

a Verify in each case that $z^3 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$.

b i Reduce the arguments of z_1, z_2, z_3, z_4 and z_5 to lie in the range $(-\pi, \pi]$.

ii If asked for $z \in C$ such that $z^3 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$, what would be a complete answer?

iii Illustrate the set of solutions on an Argand diagram.

Answers to AH Maths (MiA), pg. 218-9, Ex. 12.6

1 c i $-2 + 2i$ ii $-8i$ iii -64

2 a $27 \operatorname{cis} \frac{3\pi}{5}$ b $256 \operatorname{cis} \left(-\frac{2\pi}{3}\right)$

c $\operatorname{cis} 0 = 1$

d $\operatorname{cis} \frac{4\pi}{7}$

4 a $\operatorname{cis} 40^\circ$ b $\operatorname{cis} (-165^\circ)$

c $\operatorname{cis} 5^\circ$

d $\operatorname{cis} 170^\circ$

e $\operatorname{cis} 20^\circ$

f $\operatorname{cis} (-6^\circ)$

g $\operatorname{cis} 120^\circ$

h $\operatorname{cis} 175^\circ$

i $\operatorname{cis} 100^\circ$

j $\operatorname{cis} 24^\circ$

k $\operatorname{cis} 90^\circ$

l $\operatorname{cis} 88^\circ$

6 a i $\cos^3 \theta + 3 \cos^2 \theta \sin \theta i - 3 \cos \theta \sin^2 \theta - \sin^3 \theta i$

ii $\cos 3\theta + i \sin 3\theta$

b i $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$

ii $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

c $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

d $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$

7 a i $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$

ii $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$

iii $\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 8 \cos^2 \theta - 1)$

9 a Verifications

b i $\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}, -\frac{5\pi}{12}$

ii $\operatorname{cis} \left(\frac{11\pi}{12}\right), \operatorname{cis} \left(-\frac{5\pi}{12}\right), \operatorname{cis} \left(\frac{\pi}{4}\right)$

iii

