30 / 1 / 18

Matrices and Systems of Equations - Lesson 3

Basic Matrix Algebra

LI

- Add, subtract and Scalar Multiply matrices.
- Transpose of a matrix.
- Symmetric and Skew-Symmetric matrices.
- Matrix Properties 1.

SC

- Primary school arithmetic.

An $m \times n$ ('m by n') matrix (plural : matrices) is a rectangular arrangement of (usually) numbers (entries) with m rows and n columns

A matrix is usually denoted by an uppercase letter and the entries usually by lowercase letters. If A is an $m \times n$ matrix (or, a matrix of order $m \times n$), we can write A alternatively as :

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

A_{ij} is the $(i, j)^{th}$ entry of A

$$(1 \leq i \leq m, 1 \leq j \leq n)$$

A matrix is square if it has the same number of rows as columns; the main diagonal of such an $n \times n$ matrix consists of all entries of the form A_{ii} $(1 \leq i \leq n)$

The zero matrix $O_{m,n}$ is the $m \times n$ matrix consisting entirely of zeroes

The identity matrix of order n is the $n \times n$ matrix I_n consisting of the number 1 down the main diagonal, with all other entries 0

Two matrices are equal if they have the same order and all corresponding entries are equal

Two or more matrices can be added or subtracted only if they have the same order; in particular, the sum/difference of matrices A and B (both of order $m \times n$) is the $m \times n$ matrix obtained by adding/subtracting corresponding entries of A and B :

$$(A \pm B)_{ij} = A_{ij} \pm B_{ij}$$

The scalar multiple of an $m \times n$ matrix A is the $m \times n$ matrix obtained by multiplying each entry of A by the same real number k :

$$(k A)_{ij} = k A_{ij}$$

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T (or A') given by interchanging the rows and columns of A :

$$(A^T)_{ij} = A_{ji}$$

A matrix A is symmetric if it equals its transpose :

$$A \text{ symmetric} \longleftrightarrow A = A^T$$

A matrix A is skew-symmetric (aka antisymmetric) if it equals the negative of its transpose :

$$A \text{ skew-symmetric} \longleftrightarrow A = -A^T$$

Symmetric and skew-symmetric matrices are necessarily square

Matrix Properties - 1

1) $A + B = B + A$

2) $(A + B) + C = A + (B + C)$

3) $A + O = O + A = A$

4) $k(A + B) = kA + kB$

5) $(A + B)^T = A^T + B^T$

6) $(A^T)^T = A$

7) $(kA)^T = kA^T$

Example 1

Simplify $2\begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$.

$$2\begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 6 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 6 & 12 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -5 & 6 \\ 2 & 0 \end{pmatrix}}$$

Example 2

Solve for S :

$$\begin{pmatrix} -3 & 1 & 2 \\ 7 & 0 & -3 \end{pmatrix} - S = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$$

$$S = \begin{pmatrix} -3 & 1 & 2 \\ 7 & 0 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$$

$$\boxed{S = \begin{pmatrix} -4 & -3 & 4 \\ 7 & -1 & -9 \end{pmatrix}}$$

Example 3

Find the value of $3u + 2w$ if :

$$\begin{pmatrix} 2u + 5w \\ 6w + 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 13 \end{pmatrix}$$

Equating corresponding entries gives,

$$2u + 5w = -8$$
$$6w + 1 = 13$$

The second equation gives $w = 2$; substituting this back into the first equation gives $u = -9$.

$$3u + 2w = 3(-9) + 2(2)$$

$$\boxed{3u + 2w = -23}$$

Sep 12-14:22

Example 4

Find A^T if $A = \begin{pmatrix} 3 & -4 \\ 0 & 1 \\ 8 & 9 \end{pmatrix}$; state the orders of A and A^T.

Interchanging rows and columns of A gives,

$$\boxed{A^T = \begin{pmatrix} 3 & 0 & 8 \\ -4 & 1 & 9 \end{pmatrix}}$$

$\boxed{\text{Order of } A : 3 \times 2}$

$\boxed{\text{Order of } A^T : 2 \times 3}$

Sep 12-14:22

Example 5

Prove that $(A + B)^T = A^T + B^T$.

Considering the $(i, j)^{th}$ entry of $(A + B)^T$, we have,

$$\begin{aligned}((A + B)^T)_{ij} &= (A + B)_{ji} \\ &= A_{ji} + B_{ji} \\ &= (A^T)_{ij} + (B^T)_{ij}\end{aligned}$$

$$\therefore \quad \boxed{(A + B)^T = A^T + B^T}$$

Sep 12-14:22

AH Maths - MiA (2nd Edn.)

- pg. 231-2 Ex. 13.1
  Q 2 - 4, 6 g - t, 7 - 10.

Ex. 13.1

2 Write the order of each of these matrices.

a $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ b $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ c $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

d $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \\ 8 & 6 & -1 & 3 \end{pmatrix}$ e $\begin{pmatrix} 1 & -3 & 0 \end{pmatrix}$ f $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 4 & 5 & 8 \end{pmatrix}$

3 Find the value of x in each case.

a $\begin{pmatrix} 2 \\ 2x \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ b $\begin{pmatrix} 2x \\ x \\ -x \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$ c $\begin{pmatrix} 0 & 4x+1 & 2x-3 \\ 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 21 & 7 \\ 1 & 1 & 5 \end{pmatrix}$

4 Find the values of x and y in each case.

a $\begin{pmatrix} 3x \\ 12 \end{pmatrix} = \begin{pmatrix} 6 \\ y \end{pmatrix}$ b $(x-y \;\; y) = (8 \;\; 7)$ c $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} x & 1-y \\ 4 & 2x \end{pmatrix}$

d $\begin{pmatrix} x+y \\ 2x \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$ e $\begin{pmatrix} 2x+3y \\ x+2y \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ f $\begin{pmatrix} 2 & 3x+y \\ 4x-3y & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 17 & 5 \end{pmatrix}$

6 Simplify

g $\begin{pmatrix} 3 & 0 \\ 0 & 5 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 5 \end{pmatrix} + \begin{pmatrix} -4 & 2 \\ 3 & -3 \\ 4 & -9 \end{pmatrix}$ h $\begin{pmatrix} 3 & 4 & 2 \\ 2 & 1 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 1 & -1 \\ -3 & 0 & 4 \end{pmatrix}$ i $\begin{pmatrix} 4 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & -3 \end{pmatrix}$

j $\begin{pmatrix} 5 & -3 & 2 \\ -2 & -1 & 3 \\ 0 & 2 & -2 \end{pmatrix} + \begin{pmatrix} -5 & 3 & -2 \\ 2 & 1 & -3 \\ 0 & -2 & 2 \end{pmatrix}$ k $\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ l $\begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

m $\begin{pmatrix} a+2b \\ 2a-b \end{pmatrix} - \begin{pmatrix} a+b \\ 3a-b \end{pmatrix}$ n $(4a \;\; 2b) - (-a \;\; -b)$

o $(3m \;\; -2n) + (-m \;\; 4n) - (5m \;\; 3n)$ p $\begin{pmatrix} 2 & 5 \\ 1 & -4 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}$

q $\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} -3 & 1 \\ 3 & -3 \\ 3 & -1 \end{pmatrix}$ r $5\begin{pmatrix} 1 & -4 & -6 \\ 3 & -3 & -1 \end{pmatrix}$

s $-3\begin{pmatrix} -1 & 3 & 1 \\ -4 & 2 & 0 \\ -5 & -1 & -3 \end{pmatrix}$ t $3\begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & -2 \\ 5 & 1 & 1 \end{pmatrix} - 2\begin{pmatrix} 3 & -4 & -1 \\ 3 & 2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$

7 A, B, C, D, E and F are matrices. Solve these equations to find them.

a $A + \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 5 \end{pmatrix}$ b $\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} + B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

c $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} + C = \begin{pmatrix} 4 & 7 \\ 1 & 3 \end{pmatrix}$ d $D - \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 6 \end{pmatrix}$

e $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} - E$ f $F - \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 9 & 8 \end{pmatrix}$

8 Given $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 1 & 2 \\ -5 & 0 & 2 \end{pmatrix}$ evaluate

a $2A + 3B - C$ b $3A - 2B + C$ c $A - 2B + 3C$ d $2(A + B + C)$

9 Solve these matrix equations.

a $3A = \begin{pmatrix} 6 & -3 \\ 0 & 9 \end{pmatrix}$ b $2B + \begin{pmatrix} 1 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 9 \end{pmatrix}$ c $2C - 3\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 1 \\ 6 & -2 & 3 \end{pmatrix}$

10 a Write the transpose of each of these matrices.

i $\begin{pmatrix} 3 & 4 & 1 \\ -7 & 9 & 5 \end{pmatrix}$ ii $\begin{pmatrix} 3 & -6 & 0 \\ -4 & 7 & 1 \\ 7 & 3 & -2 \end{pmatrix}$ iii $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ iv $\begin{pmatrix} 0 & 1 & -5 \\ -1 & 0 & 2 \\ 5 & -2 & 0 \end{pmatrix}$

b Comment on your answers for iii and iv.

Answers to AH Maths (MiA), pg. 231-2, Ex. 13.1

2 **a** 2×2 **b** 3×1 **c** 2×3
d 3×4 **e** 1×3 **f** 3×3

3 **a** 3 **b** 3 **c** 5

4

	a	b	c	d	e	f
x	2	15	1	4	1	2
y	12	7	−2	3	2	−3

6 **f**

g $\begin{pmatrix} 1 & 5 \\ 4 & 6 \\ 13 & -3 \end{pmatrix}$

h $\begin{pmatrix} 5 & 5 & 1 \\ -1 & 1 & 10 \end{pmatrix}$ **i** $\begin{pmatrix} 6 & 1 & 5 \\ 3 & 0 & 0 \\ -2 & 3 & -4 \end{pmatrix}$

j $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **k** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

l $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ **m** $\begin{pmatrix} b \\ -a \end{pmatrix}$

n $(5a \;\; 3b)$ **o** $(-3m \;\; -n)$

p $\begin{pmatrix} 4 & 1 \\ -2 & -5 \\ -5 & 4 \end{pmatrix}$ **q** $\begin{pmatrix} 5 & 1 \\ -3 & 3 \\ -1 & 3 \end{pmatrix}$

r $\begin{pmatrix} 5 & -20 & -30 \\ 15 & -15 & -5 \end{pmatrix}$ **s** $\begin{pmatrix} 3 & -9 & -3 \\ 12 & -6 & 0 \\ 15 & 3 & 9 \end{pmatrix}$

t $\begin{pmatrix} 6 & 11 & -4 \\ -9 & -4 & -6 \\ 13 & 3 & 9 \end{pmatrix}$

7 **a** $\begin{pmatrix} 3 & 6 \\ -1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$

c $\begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix}$ **d** $\begin{pmatrix} 5 & 4 \\ 5 & 10 \end{pmatrix}$

e $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \end{pmatrix}$ **f** $\begin{pmatrix} 3 & 4 \\ 7 & 6 \\ 14 & 15 \end{pmatrix}$

8 **a** $\begin{pmatrix} -2 & 0 & -5 \\ 18 & 8 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 7 & -4 & 4 \\ -5 & -1 & 6 \end{pmatrix}$

c $\begin{pmatrix} 13 & 0 & 8 \\ -19 & -3 & 6 \end{pmatrix}$ **d** $\begin{pmatrix} 10 & 2 & 2 \\ 0 & 6 & 10 \end{pmatrix}$

9 **a** $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$

c $\begin{pmatrix} 5 & 4 & \frac{7}{2} \\ 6 & -\frac{5}{2} & \frac{3}{2} \end{pmatrix}$

10 **a** **i** $\begin{pmatrix} 3 & -7 \\ 4 & 9 \\ 1 & 5 \end{pmatrix}$ **ii** $\begin{pmatrix} 3 & -4 & 7 \\ -6 & 7 & 3 \\ 0 & 1 & -2 \end{pmatrix}$

iii $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ **iv** $\begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & -2 \\ -5 & 2 & 0 \end{pmatrix}$

b **iii** Symmetric **iv** Skew-symmetric

Jun 1-11:51