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Unit 1 : Differential Equations - Lesson 3

2^{nd}-Order Homogeneous Differential Equations

LI

- Solve DEs of the form $ay'' + by' + cy = 0$.

SC

- Auxiliary Equation.

A 2nd-order (linear, ordinary) homogeneous differential equation (with constant coefficients) is a differential equation that can be written in the form :

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \bigstar$$

$$(a y'' + b y' + c y = 0)$$

$$(a, b, c \in \mathbb{R})$$

Solving the above type of differential equation requires the following steps :

- Form the associated Auxiliary Equation :

$$a m^2 + b m + c = 0 \quad (m \in \mathbb{C})$$

- Solve the Auxiliary Equation for m; there are 3 possibilities :

Solution(s) for m	General solution of $\bigstar$
2 Real, distinct (m, n)	$y_G = A e^{mx} + B e^{nx}$
1 Real, repeated (m)	$y_G = A e^{mx} + B x e^{mx}$
2 Complex conjugates $(m = p + ri, \overline{m} = p - ri)$	$y_G = e^{px}(A \cos rx + B \sin rx)$

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Example 1

Obtain the general solution of,

$$y'' - 7y' + 10y = 0$$

The Auxiliary Equation is,

$$m^2 - 7m + 10 = 0$$

Solving this for m gives,

$$(m - 2)(m - 5) = 0$$

$$\Rightarrow \qquad \underline{m = 2, m = 5}$$

The general solution is thus,

$$\boxed{y_G = Ae^{2x} + Be^{5x}}$$

Example 2

Obtain the particular solution of,

$$y'' - 7y' + 10y = 0$$

with the initial conditions $y = -1$ and $y' = -11$ when $x = 0$.

From Example 1, the general solution is,

$$y_G = A e^{2x} + B e^{5x}$$

Differentiating this gives,

$$y_G' = 2 A e^{2x} + 5 B e^{5x}$$

The initial conditions respectively give,

$$-1 = A + B$$

$$-11 = 2A + 5B$$

Solving these simultaneous equations gives,

$$\underline{A = 2, B = -3}$$

The required particular solution is thus,

$$\boxed{y = 2e^{2x} - 3e^{5x}}$$

Example 3

Obtain the general solution of,

$$4y'' - 4y' + y = 0$$

Solving the Auxiliary Equation gives,

$$4m^2 - 4m + 1 = 0$$

$$\Rightarrow \quad (2m - 1)(2m - 1) = 0$$

$$\Rightarrow \quad \underline{m = 1/2}$$

$$\therefore \quad \boxed{y_G = Ae^{x/2} + Bxe^{x/2}}$$

Example 4

Obtain the particular solution of,

$$4y'' - 4y' + y = 0$$

satisfying $y(0) = 3$ and $y(2) = e$.

From Example 3, the general solution is,

$$y_G(x) = A e^{x/2} + B x e^{x/2}$$

The first initial condition gives,

$$\underline{3 = A}$$

The second initial condition then gives,

$$e = A e + 2 B e$$

$\Rightarrow$ $$e = 3 e + 2 B e$$

$\Rightarrow$ $$\underline{B = -1}$$

The required particular solution is thus,

$$\boxed{y = 3 e^{x/2} - x e^{x/2}}$$

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Example 5

Obtain the general solution of,

$$y'' - 4y' + 29y = 0$$

The Auxiliary Equation is,

$$m^2 - 4m + 29 = 0$$

The Quadratic Formula gives,

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)}$$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{-100}}{2}$$

$$\Rightarrow \quad m = \frac{4 \pm 10i}{2}$$

$$\Rightarrow \quad m = 2 \pm 5i$$

$$\therefore \quad \boxed{y = e^{2x}(A\cos 5x + B\sin 5x)}$$

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Example 6

Obtain the particular solution of,

$$y'' - 4y' + 29y = 0$$

satisfying $y(0) = 1$ and $y'(0) = -2.$

From Example 5, the general solution is,

$$y_G(x) = e^{2x}(A\cos 5x + B\sin 5x)$$

Differentiating this (using the Product Rule) gives,

$$y_G'(x) = 2e^{2x}(A\cos 5x + B\sin 5x) + e^{2x}(-5A\sin 5x + 5B\cos 5x)$$

The initial conditions respectively give,

$$\underline{1 = A}$$

$$-2 = 2A + 5B$$

$$\Rightarrow \quad \underline{B = -4/5}$$

The required particular solution is thus,

$$\boxed{y(x) = e^{2x}(\cos 5x - (4/5)\sin 5x)}$$

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AH Maths - MiA (2nd Edn.)

- pg. 140 Ex. 8.4 Q 1 b, h,
  2 a, d, e.
- pg. 141 Ex. 8.5 Q 1 f, g, h,
  2 a, c.
- pg. 142 Ex. 8.6 Q 1 d, f, 2 a, d.

Ex. 8.4

1 Find the general solution in each case.

b $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 5y = 0$

h $3\dfrac{d^2y}{dx^2} + 13\dfrac{dy}{dx} + 4y = 0$

2 Find the particular solution in each case given the initial conditions.

a $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ when $x = 0, y = 4$ and $\dfrac{dy}{dx} = 10$

d $9\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} + 2y = 0$ when $x = 0, y = 4$ and $\dfrac{dy}{dx} = -2$

e $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0$ when $x = 0, y = 3$ and when $x = \frac{1}{4}, y = 1 + 2e$

Ex. 8.5

1 Find the general solution in each case.

f $16\frac{d^2y}{dx^2} + 40\frac{dy}{dx} + 25y = 0$

g $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

h $\frac{d^2y}{dx^2} = 0$

2 Find the particular solution in each case given the initial conditions.

a $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$ when $x = 0, y = 1$ and when $x = 1, y = 3e^{-6}$

c $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ when $x = 0, y = 2$ and $\frac{dy}{dx} = \frac{7}{3}$

Ex. 8.6

1 Find the general solution in each case.

d $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$

f $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 101y = 0$

2 Find the particular solution in each case given the initial conditions.

a $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$ when $x = 0, y = 2$ and $\frac{dy}{dx} = 9$

d $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 82y = 0$ when $x = 0, y = 3$ and $\frac{dy}{dx} = 16$

Answers to AH Maths (MiA), pg. 140, Ex. 8.4

1

b $y = Ae^{5x} + Be^{x}$

h $y = Ae^{-\frac{x}{3}} + Be^{-4x}$

2

a $y = 2e^{3x} + 2e^{2x}$

d $y = 2e^{-\frac{x}{3}} + 2e^{-\frac{2x}{3}}$

e $y = 2e^{4x} + 1$

Answers to AH Maths (MiA), pg. 141, Ex. 8.5

1

f $y = Ae^{-\frac{5x}{4}} + Bxe^{-\frac{5x}{4}}$

g $y = Ae^{5x} + Bxe^{5x}$

h $y = Ax + B$

2

a $y = e^{-6x} + 2xe^{-6x}$

c $y = 2e^{\frac{2x}{3}} + xe^{\frac{2x}{3}}$

Answers to AH Maths (MiA), pg. 142, Ex. 8.6

1

d $y = e^{-3x}(A\cos 2x + B\sin 2x)$

f $y = e^{-0.5x}(A\cos 5x + B\sin 5x)$

2

a $y = e^{3x}(2\cos x + 3\sin x)$

d $y = e^{\frac{x}{3}}(3\cos 3x + 5\sin 3x)$