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*Unit 2 : The Binomial Theorem - Lesson 2*

## The Binomial Theorem

### LI

- Know the Binomial Theorem.
- Use the Binomial Theorem to solve problems.

### SC

- Binomial coefficients.
- Pascal's Triangle.
- General Term.

## Pascal's Triangle and Binomial Coefficients

**Pascal's Triangle** is the following arrangement of numbers :

						Column $r = 0$				
						Column $r = 1$				
Row $n = 0$					1	Column $r = 2$				
Row $n = 1$					1	1	Column $r = 3$			
Row $n = 2$					1	2	1	Column $r = 4$		
Row $n = 3$					1	3	3	1	Column $r = 5$	
Row $n = 4$					1	4	6	4	1	
Row $n = 5$					1	5	10	10	5	1
					.					
					.					
					.					

The binomial coefficient  ${}^n C_r$  is the number in Pascal's Triangle that is located in row  $n$  and column  $r$ .

The construction of Pascal's Triangle explains the identity  ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$ ; the number in row  $n + 1$  and column  $r$  is obtained by adding the numbers in row  $n$  and columns  $r$  and  $r - 1$ .

A **binomial** is an expression involving two terms

The **Binomial Theorem** is a quick way of taking powers of a binomial :

$$(x + y)^n = \sum_{r=0}^n {}^n C_r x^{n-r} y^r = \sum_{r=0}^n {}^n C_r x^r y^{n-r}$$

The **General Term (GT)** in the above Binomial Theorem is :

$$GT = {}^n C_r x^{n-r} y^r = {}^n C_r x^r y^{n-r}$$

Example 1

Expand  $(a + b)^5$  using the Binomial Theorem.

$$(a + b)^5 = \sum_{r=0}^5 {}^5C_r a^{5-r} b^r$$

$$\begin{aligned} \therefore (a + b)^5 &= {}^5C_0 a^5 b^0 + {}^5C_1 a^4 b^1 + {}^5C_2 a^3 b^2 \\ &\quad + {}^5C_3 a^2 b^3 + {}^5C_4 a^1 b^4 + {}^5C_5 a^0 b^5 \end{aligned}$$

Using Pascal's Triangle for  $n = 5$  allows us to evaluate the binomial coefficients. Hence,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example 2

Find the binomial expansion of  $(2 - 3p)^4$ .

$$(2 - 3p)^4 = \sum_{r=0}^4 {}^4C_r 2^{4-r} (-3p)^r$$

$$\begin{aligned} \therefore (2 - 3p)^4 &= {}^4C_0 2^4 (-3p)^0 + {}^4C_1 2^3 (-3p)^1 \\ &\quad + {}^4C_2 2^2 (-3p)^2 + {}^4C_3 2^1 (-3p)^3 \\ &\quad + {}^4C_4 2^0 (-3p)^4 \end{aligned}$$

$$\begin{aligned} \Rightarrow (2 - 3p)^4 &= 1 \cdot 16 \cdot 1 + 4 \cdot 8 \cdot (-3p) + 6 \cdot 4 \cdot 9p^2 \\ &\quad + 4 \cdot 2 \cdot (-27p^3) + 1 \cdot 1 \cdot 81p^4 \end{aligned}$$

$$\Rightarrow (2 - 3p)^4 = 16 - 96p + 216p^2 - 216p^3 + 81p^4$$

Example 3

Write down and simplify the general term of  $\left(x^2 - \frac{3}{x}\right)^{16}$ .

Hence obtain the coefficient of  $x^{23}$ .

$$\left(x^2 - \frac{3}{x}\right)^{16} = \sum_{r=0}^{16} {}^{16}C_r (x^2)^{16-r} \left(-\frac{3}{x}\right)^r$$

$$GT = {}^{16}C_r (x^2)^{16-r} \left(-\frac{3}{x}\right)^r$$

$$\therefore GT = {}^{16}C_r \cdot x^{32-2r} \cdot (-3)^r \cdot x^{-r}$$

$$\Rightarrow GT = {}^{16}C_r (-1)^r 3^r x^{32-3r}$$

The coefficient of  $x^{23}$  is obtained when,

$$32 - 3r = 23$$

$$\Rightarrow 3r = 9$$

$$\Rightarrow \underline{r = 3}$$

Hence,

$$\text{Coefficient of } x^{23} = {}^{16}C_3 \cdot (-1)^3 3^3$$

$$\Rightarrow \text{Coefficient of } x^{23} = -15120$$

Example 4

Find the term independent of  $x$  in  $\left(2x^2 + \frac{1}{3x}\right)^6$ .

$$\left(2x^2 + \frac{1}{3x}\right)^6 = \sum_{r=0}^6 {}^6C_r (2x^2)^{6-r} \left(\frac{1}{3x}\right)^r$$

$$GT = {}^6C_r (2x^2)^{6-r} \left(\frac{1}{3x}\right)^r$$

$$\therefore GT = {}^6C_r \cdot 2^{6-r} x^{12-2r} \cdot 3^{-r} \cdot x^{-r}$$

$$\Rightarrow \underline{GT = {}^6C_r 2^{6-r} 3^{-r} x^{12-3r}}$$

The term independent of  $x$  occurs when,

$$12 - 3r = 0$$

$$\Rightarrow 3r = 12$$

$$\Rightarrow \underline{r = 4}$$

Hence,

$$\text{Term independent of } x = {}^6C_4 2^{6-4} 3^{-4}$$

$$\Rightarrow \boxed{\text{Term independent of } x = \frac{20}{27}}$$

Example 5

Find the coefficient of  $x^5$  in  $(1 + x)^4(1 - 2x)^3$ .

The general term of  $(1 + x)^4$  is  ${}^4C_r x^r$ ; the general term of  $(1 - 2x)^3$  is  ${}^3C_p (-2x)^p$ .

The general term of  $(1 + x)^4(1 - 2x)^3$  is thus,

$$\begin{aligned} & {}^4C_r x^r \cdot {}^3C_p (-2x)^p \\ &= {}^4C_r (-2)^p x^r {}^3C_p x^p \\ &= \underline{{}^4C_r {}^3C_p (-2)^p x^{r+p}} \end{aligned}$$

We require  $r + p = 5$ ; the possibilities for  $r$  and  $p$  are thus :

$$r = 2, p = 3$$

$$r = 3, p = 2$$

$$r = 4, p = 1$$

Hence, the coefficient of  $x^5$  is,

$$\begin{aligned} & {}^4C_2 {}^3C_3 (-2)^3 + {}^4C_3 {}^3C_2 (-2)^2 + {}^4C_4 {}^3C_1 (-2)^1 \\ &= 6 \cdot 1 \cdot (-8) + 4 \cdot 3 \cdot 4 + 1 \cdot 3 \cdot (-2) \\ &= \boxed{-6} \end{aligned}$$

Example 6

By considering the expansion of  $(1 - x)^4$ , find  $(0.98)^4$ .

$$(1 - x)^4 = \sum_{r=0}^4 {}^4C_r 1^{4-r} (-x)^r$$

$$\begin{aligned} \therefore (1 - x)^4 &= {}^4C_0 (-x)^0 + {}^4C_1 (-x)^1 + {}^4C_2 (-x)^2 \\ &\quad + {}^4C_3 (-x)^3 + {}^4C_4 (-x)^4 \end{aligned}$$

$$\underline{(1 - x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4}$$

Taking  $x = 0.02$ ,

$$\begin{aligned} (0.98)^4 &= 1 - 4(0.02) + 6(0.02)^2 - 4(0.02)^3 \\ &\quad + (0.02)^4 \end{aligned}$$

$$\begin{aligned} \Rightarrow (0.98)^4 &= 1 - 0.08 + 0.0024 - 0.000032 \\ &\quad + 0.00000016 \end{aligned}$$

$$\Rightarrow (0.98)^4 = 0.92236816$$

## AH Maths - MiA (2<sup>nd</sup> Edn.)

- pg. 36-7 Ex. 3.4 Q 1, 2 a, 4.
- pg. 38-9 Ex. 3.5 Q 1, 7.
- pg. 40 Ex. 3.6 Q 1 a, b, c, d.

Ex. 3.4
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**1** Use the binomial theorem to expand these expressions.

**a**  $(a + b)^5$

**b**  $(1 + 2x)^3$

**c**  $(2 + 3b)^4$

**d**  $(3a + 2b)^3$

**e**  $(a - b)^4$

**f**  $(1 - p)^3$

**g**  $(3 - x)^4$

**h**  $(2a - 3b)^3$

**2 a** Expand these expressing your answer as positive powers of  $x$ .

**i**  $\left(x + \frac{1}{x}\right)^3$

**ii**  $\left(x + \frac{1}{x}\right)^4$

**iii**  $\left(x - \frac{1}{x}\right)^5$

**iv**  $\left(x - \frac{1}{x}\right)^6$

**4** Calculate the term

**a** containing  $x^4$  in the expansion of  $(x + y)^8$

**b** containing  $a^3$  in the expansion of  $(3 + 2a)^5$

**c** whose coefficient is 64 in the expansion of  $(2 + x)^6$

**d** containing  $x^3$  in the expansion of  $(x - 7)^5$

**e** containing  $a^4$  term in the expansion of  $(1 - 3a)^6$

**f** independent of  $x$  in the expansion of  $\left(x + \frac{1}{x}\right)^8$ .

Ex. 3.5
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**1** What is the coefficient of

**a**  $x^4$  in the expansion  $(1 + x)^2(1 + 2x)^3$

**b**  $x^5$  in the expansion  $(1 - x)^3(2 + x)^4$

**c**  $x^7$  in the expansion  $(1 + 2x)^4(1 - 2x)^6$ ?

**7** Find the term independent of  $a$  in  $\left(\frac{3}{2}a^2 - \frac{1}{3a}\right)^9$ .

Ex. 3.6
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**1** Calculate these powers correct to 3 significant figures.

**a**  $1.01^5$

**b**  $1.04^6$

**c**  $0.94^7$

**d**  $12.01^5$

## Answers to AH Maths (MiA), pg. 36-7, Ex. 3.4

- 1 a**  $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
- b**  $1^3 + 3 \cdot 1^2 \cdot (2x) + 3 \cdot 1 \cdot (2x)^2 + (2x)^3 = 1 + 6x + 12x^2 + 8x^3$
- c**  $16 + 96b + 216b^2 + 216b^3 + 81b^4$
- d**  $27a^3 + 54a^2b + 36ab^2 + 8b^3$
- e**  $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
- f**  $1 - 3p + 3p^2 - p^3$
- g**  $81 - 108x + 54x^2 - 12x^3 + x^4$
- h**  $8a^3 - 36a^2b + 54ab^2 - 27b^3$
- 2 a**  $x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$
- ii**  $x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 =$   
 $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$
- iii**  $x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$
- iv**  $x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$
- 4 a**  $70x^4y^4$       **b**  $720a^3$       **c**  $64x^0$
- d**  $490x^3$       **e**  $1215a^4$       **f**  $70$

Answers to AH Maths (MiA), pg. 38-9, Ex. 3.5

1 a 28                      b -3                      c 1024

7  $\frac{7}{18}$

Answers to AH Maths (MiA), pg. 40, Ex. 3.6

1 a 1.05                      b 1.27  
c 0.648                      d 250 000