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Unit 2 : Arithmetic and Algebra of Complex Numbers - Lesson 2

Solving Complex Quadratics and Other Simple Equations

LI

- Solve complex quadratics of the form $a z^2 + b z + c = 0$.
- Solve complex equations of the form $z^2 = a + b i$.
- Solve other simple equations.

SC

- Quadratic formula.
- Equating real and imaginary parts.

Example 1

Solve $z^2 - 2z + 5 = 0$ for z.

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow \quad z = \frac{2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow \quad z = \frac{2 \pm \sqrt{(16)(-1)}}{2}$$

$$\Rightarrow \quad z = \frac{2 \pm 4\sqrt{-1}}{2}$$

$$\Rightarrow \quad z = \frac{2 \pm 4i}{2}$$

$$\Rightarrow \quad \boxed{z = 1 \pm 2i}$$

Example 2

Solve $z^2 + 3z + 10 = 0$ for z.

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(10)}}{2(1)}$$

$$\Rightarrow \quad z = \frac{-3 \pm \sqrt{-31}}{2}$$

$$\Rightarrow \quad \boxed{z = -\frac{3}{2} \pm \frac{\sqrt{31}}{2}\, i}$$

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Example 3

Find two numbers that have a sum of 12 and a product of 40.

Let the two numbers be a and b. We require,

$$a + b = 12$$
$$ab = 40$$

Solving the second equation for b and substituting into the first equation gives,

$$a + \frac{40}{a} = 12$$

$$\Rightarrow \quad a^2 + 40 = 12a$$

$$\Rightarrow \quad a^2 - 12a + 40 = 0$$

This is a quadratic in a. Solving it gives (check !),

$$\underline{a = 6 + 2i, 6 - 2i}$$

The first equation ($a + b = 12$) then gives the corresponding solutions for b :

$$b = 12 - (6 + 2i) \Rightarrow \underline{b = 6 - 2i}$$

$$b = 12 - (6 - 2i) \Rightarrow \underline{b = 6 + 2i}$$

So, $a = 6 + 2i \Rightarrow b = 6 - 2i$ (and vice versa); the numbers are thus,

$$\boxed{6 + 2i, 6 - 2i}$$

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Example 4

Find the square roots of $5 + 12i$.

Let $a + bi$ be a square root of $5 + 12i$, i.e.,

$$a + bi = \sqrt{5 + 12i}$$

$$\therefore \quad (a + bi)^2 = 5 + 12i$$

$$\Rightarrow \quad (a + bi)(a + bi) = 5 + 12i$$

$$\Rightarrow \quad (a^2 - b^2) + (2ab)i = 5 + 12i$$

Equating real and imaginary parts gives,

$$a^2 - b^2 = 5$$

$$2ab = 12$$

Solving the second equation for b ($b = 6/a$) and substituting into the first equation gives,

$$a^2 - \frac{36}{a^2} = 5$$

$$\Rightarrow \quad a^4 - 36 = 5a^2$$

$$\Rightarrow \quad a^4 - 5a^2 - 36 = 0$$

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This is a quadratic in the variable a^2. Factorising gives,

$$(a^2 - 9)(a^2 + 4) = 0$$

$$\Rightarrow \quad a^2 - 9 = 0, \; a^2 + 4 = 0$$

The second equation ($a^2 = -4$) has no solutions for a (as a is a real number). So,

$$\underline{a = 3} \Rightarrow b = 6/3 \Rightarrow \underline{b = 2}$$

$$\underline{a = -3} \Rightarrow b = 6/(-3) \Rightarrow \underline{b = -2}$$

$$\therefore \quad \boxed{\sqrt{5 + 12i} = 3 + 2i, -3 - 2i}$$

Example 5

Solve for z,

$$z + 2\bar{z} = 6 + 8i$$

Let $z = x + yi$. Then,

$$x + yi + 2(x - yi) = 6 + 8i$$

$$\Rightarrow \quad x + 2x + yi - 2yi = 6 + 8i$$

$$\Rightarrow \quad 3x + (-y)i = 6 + 8i$$

Equating real and imaginary parts gives,

$$3x = 6 \Rightarrow \underline{x = 2}$$

$$-y = 8 \Rightarrow \underline{y = -8}$$

$$\therefore \quad \boxed{z = 2 - 8i}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 207-8 Ex. 12.1 Q 3, 5.
- pg. 209 Ex. 12.2 Q 4, 5.

Solve for z :

1) $z - 3\bar{z} = 9 + 12i.$

2) $3\bar{z} + iz = 10 - 18i.$

Ex. 12.1

3 Solve these quadratic equations giving the roots in the form $z = a \pm bi$.

a $z^2 + 2z + 2 = 0$ b $z^2 + 4z + 13 = 0$ c $z^2 - 6z + 13 = 0$

d $2z^2 - 4z + 10 = 0$ e $3z^2 - 12z + 15 = 0$ f $2z^2 + 12z + 36 = 0$

5 Solve Cardan's problem, namely:
Find two numbers which have a sum of 10 and a product of 40.

Ex. 12.2

4 Find a and b in each case so that $(a + ib)^2$ is equal to

a $5 - 12i$ b $15 - 8i$ c $-24 - 10i$

5 Calculate

a $\sqrt{3 - 4i}$ b $\sqrt{21 - 20i}$ c $\sqrt{-9 + 40i}$

Answers to AH Maths (MiA), pg. 207-8, Ex. 12.1

3 **a** $-1 \pm i$ **b** $-2 \pm 3i$ **c** $3 \pm 2i$

d $1 \pm 2i$ **e** $2 \pm i$ **f** $-3 \pm 3i$

5 $a + b = 10$ and $ab = 40 \Rightarrow a^2 - 10a + 40 = 0 \Rightarrow$

$a = 5 \pm \sqrt{15}\, i$

Answers to AH Maths (MiA), pg. 209, Ex. 12.2

4 **a** $-3 + 2i$ and $3 - 2i$ **b** $-4 + i$ and $4 - i$

c $-1 + 5i$ and $1 - 5i$

5 **a** $-2 + i$ and $2 - i$ **b** $-5 + 2i$ and $5 - 2i$

c $4 + 5i$ and $-4 - 5i$

Solve for z :

1) $z - 3\bar{z} = 9 + 12i$. $z = -(9/2) + 3i$

2) $3\bar{z} + iz = 10 - 18i$. $z = 6 + 8i$