Polynomials - Lesson 2

The Remainder Theorem and Synthetic Division

<u>LI</u>

- Know the Remainder Theorem.
- Find the Quotient and Remainder of a polynomial.

<u>SC</u>

- Synthetic Division.

When a polynomial $f(x)$ is divided by $(x - h)$, it can be written as :

$$\underbrace{f(x)}_{\text{Dividend}} = \overbrace{Q(x)}^{\text{Quotient}} \underbrace{(x - h)}_{\text{Divisor}} + \overbrace{R}^{\text{Remainder}}$$

<u>Remainder Theorem</u> When a polynomial $f(x)$ is divided by $(x - h)$, the remainder is $f(h)$.

<u>Proof :</u>

When the polynomial $f(x)$ is divided by $(x - h)$, we have :

$$f(x) = Q(x)(x - h) + R$$

Putting $x = h$ gives,

$$f(h) = Q(h)(h - h) + R$$

$$\Rightarrow f(h) = R$$

i. e., the remainder $R = f(h)$.

So, to work out the quotient and remainder when $f(x)$ is divided by $(x - h)$, the remainder is given by $f(h)$; so, use Horner's Method to work out R. The coefficients of the quotient (which is a polynomial) are the numbers to the left of the remainder.☆

When Horner's Method is used in this way, it is called Synthetic Division.

Proof of ☆ :

$$f(x) = a x^3 + b x^2 + c x + d$$

From previously, $R = f(h) = a h^3 + b h^2 + c h + d.$

When dividing $f(x)$ by $(x - h)$,

$$f(x) = Q(x)(x - h) + R$$

Hence,

$$a x^3 + b x^2 + c x + d = Q(x)(x - h) + a h^3 + b h^2 + c h + d$$

$$Q(x)(x - h) = a x^3 + b x^2 + c x + d - a h^3 - b h^2 - c h - d$$

$$Q(x)(x - h) = a(x^3 - h^3) + b(x^2 - h^2) + c(x - h)$$

$$Q(x)(x - h) = (x - h)[a(x^2 + x h + h^2) + b(x + h) + c]$$

As $x \neq h$,

$$Q(x) = a(x^2 + x h + h^2) + b(x + h) + c$$

$$Q(x) = a x^2 + a x h + a h^2 + b x + b h + c$$

$$Q(x) = a x^2 + (a h + b) x + (a h^2 + b h + c) \quad \dagger$$

When working out R, we have :

	x^3	x^2	x^1	x^0
h	a	b	c	d
		$a h$	$a h^2 + h p$	$a h^3 + b h^2 + c h$
	a	$a h + b$	$a h^2 + b h + c$	$a h^3 + b h^2 + c h + d = R$

Comparing this with $\dagger$ shows that the coefficients of $Q(x)$ are exactly the numbers to the left of the remainder in the above calculation.

Example 1

Find the quotient and remainder when
$f(x) = 4x^3 - 7x^2 + 11$ is divided by $(x + 2)$.

$$\boxed{\begin{array}{l} (x + 2) \\ = (x - (-2)) \\ \therefore h = -2 \end{array}}$$

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ -2 & 4 & -7 & 0 & 11 \\ & & -8 & 30 & -60 \\ \hline & 4 & -15 & 30 & \boxed{-49} \end{array}$$

$$\boxed{\begin{array}{l} \text{Quotient } (Q(x)) = 4x^2 - 15x + 30 \\ \text{Remainder } (R) = -49 \end{array}}$$

Example 2

Divide $f(x) = 5x^3 + 2x^2 + 8x - 2$ by $(x - 2)$ and write $f(x)$ in the form $Q(x)(x - h) + R$.

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ 2 & 5 & 2 & 8 & -2 \\ & & 10 & 24 & 64 \\ \hline & 5 & 12 & 32 & \boxed{62} \end{array}$$

$$Q(x) = 5x^2 + 12x + 32, \ R = 62$$

$$\therefore \boxed{f(x) = (5x^2 + 12x + 32)(x - 2) + 62}$$

When $f(x)$ is divided by $(x - p)$, the remainder is $f(p)$.

$$\boxed{\text{When } f(x) \text{ is divided by } (qx - p)\text{, the remainder is } f(p/q).}$$

Example 3

Find the quotient and remainder when
$f(x) = 4x^4 + 2x^3 - 6x^2 + 3$ is divided by
$(2x + 1)$ and express $f(x)$ in the form
$Q(x)(2x + 1) + R.$

$$
\begin{array}{r|rrrrr}
 & x^4 & x^3 & x^2 & x^1 & x^0 \\
-1/2 & 4 & 2 & -6 & 0 & 3 \\
 & & -2 & 0 & 3 & -3/2 \\
\hline
 & 4 & 0 & -6 & 3 & \boxed{3/2}
\end{array}
$$

$$f(x) = (4x^3 - 6x + 3)(x + 1/2) + 3/2$$

$$f(x) = 2(2x^3 - 3x + 3/2)(x + 1/2) + 3/2$$

$$\boxed{f(x) = (2x^3 - 3x + 3/2)(2x + 1) + 3/2}$$

$$\boxed{Q(x) = 2x^3 - 3x + 3/2, \ R = 3/2}$$

Example 4

When $x^4 - 3x^3 + kx - 5$ is divided by $(x - 3)$, the remainder is 16. Find k.

	x^4	x^3	x^2	x^1	x^0
3	1	-3	0	k	-5
		3	0	0	$3k$
	1	0	0	k	$\boxed{3k - 5}$

But $R = 16$. So,

$$3k - 5 = 16$$

$$3k = 21$$

$$\boxed{k = 7}$$

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Questions

1 Calculate the remainder when:

a $f(x) = x^3 - x^2 + x + 1$ is divided by $x - 1$

b $f(x) = 2x^3 + x^2 - 3x - 4$ is divided by $x + 2$

c $f(x) = x^4 + x^2 - 5$ is divided by $x - 3$

d $f(x) = 2x^3 - 8x + 3$ is divided by $2x - 1$

e $f(x) = 9x^3 - 6x + 1$ is divided by $3x + 1$

2 Determine the value of a given that:

a $f(x) = x^3 + ax^2 - 3x + 1$ divided by $x + 1$ has a remainder of 5

b $f(x) = x^3 - x^2 + ax - 3$ divided by $x - 3$ has a remainder of -6

c $f(x) = ax^3 + 6x - 1$ divided by $2x - 1$ has a remainder of 3.

3 $f(x) = ax^3 - 2x^2 + 5x + b$ has a remainder of 5 when it is divided by $x - 1$ and a remainder of -11 when it is divided by $x + 1$. Determine the values of a and b.

5 Determine the quotient and remainder for each polynomial division:

a $(x^3 + 5x^2 - 3x + 1) \div (x - 2)$

b $(x^3 + 2x^2 - 3x + 4) \div (x + 3)$

c $(2x^3 - x^2 + 3x + 6) \div (x + 1)$

d $(8x^3 + 4x^2 + 1) \div (2x - 1)$

Answers

1 **a** 2

b -10

c 85

d $\frac{-3}{4}$

e $\frac{8}{3}$

2 **a** 2

b -7

c 8

3 $a = 3$

$b = -1$

5 **a** $x^2 + 7x + 11$

23

b $x^2 - x$

4

c $2x^2 - 3x + 6$

0

d $4x^2 + 4x + 2$

3