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Further Proof Techniques - Lesson 2

Proof by Counterexample (aka Disproof)

LI

- Disprove a universal statement by finding a counterexample.

SC

- Logical reasoning.

A universal statement is a statement about all possible elements of a set

A counterexample is an exception to a proposed statement, i.e., it makes the proposed statement false

If a universal statement is false, it is disproven by finding a counterexample

Notation

$\mathbb{N}$ - set of all natural numbers.

$\mathbb{W}$ - set of all whole numbers.

$\mathbb{Z}$ - set of all integers.

$\mathbb{Q}$ - set of all rational numbers.

$\mathbb{R}$ - set of all real numbers (all rational and irrational numbers).

$\mathbb{C}$ - set of all complex numbers.

Example 1

Disprove the statement : $n^3 + n + 5$ is prime $(\forall\, n \in \mathbb{N})$.

Start with the smallest value for n.

$\underline{n = 1:}$ $n^3 + n + 5 = 1^3 + 1 + 5 = 7$, which is prime.

$\underline{n = 2:}$ $n^3 + n + 5 = 2^3 + 2 + 5 = 15$, which is not prime.

$\boxed{n = 2 \text{ is a counterexample; given statement is false}}$

Example 2

Disprove the statement : x irrational $\Rightarrow x^2$ rational $(\forall x \in \mathbb{R})$.

We need to find an irrational number whose square is irrational.

Consider $x = 2^{1/4}$; x is clearly irrational. Then,

$$x^2 = (2^{1/4})^2 = 2^{2/4} = 2^{1/2} = \sqrt{2}$$

which is irrational.

$\boxed{x = 2^{1/4} \text{ is a counterexample; given statement is false}}$

Example 3

Disprove the statement :

$$x, y \text{ irrational } \Rightarrow x + y \text{ irrational } (\forall \text{ irrational } x, y).$$

Consider $x = 2 + \pi$, $y = 2 - \pi$; x and y are irrational. Then,

$$x + y = 2 + \pi + 2 - \pi = 4$$

which is not irrational.

$$\boxed{\begin{array}{l} x = 2 + \pi,\ y = 2 - \pi \text{ is a counterexample;} \\ \text{given statement is false} \end{array}}$$

Example 4

Disprove the statement : the sum of two quadratic functions is always a quadratic function.

Choosing the quadratic functions in the hypothesis to be,

$$f(x) = x^2 + 2x + 1 \text{ and } g(x) = -x^2 + x$$

shows that,

$$f(x) + g(x) = x^2 + 2x + 1 + (-x^2 + x)$$

$$\Rightarrow \quad f(x) + g(x) = 3x + 1$$

which is not a quadratic function (it is linear).

$f(x) = x^2 + 2x + 1, g(x) = -x^2 + x$ is a counterexample; given statement is false

Questions

Disprove the following statements by finding a counterexample :

1) $|a + b| = |a| + |b| \quad (\forall\, a, b \in \mathbb{R})$.

2) The sum of two cubic functions is always a cubic function.

3) If the product of two 2×2 matrices is the zero matrix, then at least one of the matrices must be the zero matrix.

4) m, n both prime $\Rightarrow m^2 + n^2$ is even ($\forall$ prime numbers m, n).

5) x, y both irrational $\Rightarrow xy$ irrational ($\forall$ irrational x, y).

6) x, y both irrational $\Rightarrow x - y$ irrational ($\forall$ irrational x, y).

7) $x^3 > x^2 \quad (\forall\, x \in \mathbb{R})$.

8) If the scalar product of two 2D vectors equals zero, then at least one of the vectors must be the zero vector.

9) $\sqrt{a + b} = \sqrt{a} + \sqrt{b} \quad (\forall\, a, b \in \mathbb{R})$.

10) $\sin(x + y) = \sin x + \sin y \quad (\forall\, x, y \in \mathbb{R})$.

11) $\ln(x + y) = \ln x + \ln y \quad (\forall\, x, y \in \mathbb{R})$.

12) $\ln(xy) = (\ln x)(\ln y) \quad (\forall\, x, y \in \mathbb{R})$.

13) For any 2×2 matrices A and B, $|A + B| = |A| + |B|$.