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Unit 1 : Differential Calculus - Lesson 2

Product and Quotient Rules

LI

- Know and use the Product and Quotient Rules to differentiate.

SC

- Derivatives from Higher Maths.

Different Notations for the Derivative

If $y = f(x)$:

- Lagrange Notation

$y'(x)$ — 'y dash(ed) of x' or 'y prime of x'

$f'(x)$

y' — 'y dash(ed)' or 'y prime'

f'

- Leibniz Notation

$\frac{d}{dx}y(x)$ — 'd by dx of y of x'

$\frac{d}{dx}f(x)$

$\frac{dy}{dx}$ — 'dy by dx'

$\frac{df}{dx}$

Notational warning : $\frac{dy}{dx}$ is not a fraction

- Euler Notation

$Dy(x)$ — 'big D of y of x'

$Df(x)$

Dy — 'big D of y'

Df

Notational warning : Dy does not mean 'D times y'

- Newton's Notation (normally for time derivatives)

$\dot{y}(t)$ means $\frac{dy}{dt}$ — 'y dot of t'

$\dot{f}(t)$

$\dot{y}$ — 'y dot'

$\dot{f}$

We mainly use Lagrange (dash) and Leibniz (d by dx) notation.

Product Rule

Used to differentiate functions that are multiplied together.

$$\frac{d}{dx}(f\,g) = \frac{df}{dx}\,g + f\,\frac{dg}{dx}$$

$$D(f\,g) = Df\,.\,g + f\,.\,Dg$$

$$(f\,g)' = f'\,g + f\,g'$$

Example 1

Differentiate $y = (2x + 3)^4 (3x - 7)^5$, fully simplifying your answer.

$$f(x) = (2x + 3)^4 \quad , \quad g(x) = (3x - 7)^5$$

$$f'(x) = 8(2x + 3)^3 \quad , \quad g'(x) = 15(3x - 7)^4$$

$$y = f(x)\,g(x)$$

$$\therefore \quad y' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\Rightarrow \quad y' = 8(2x + 3)^3 \,.\, (3x - 7)^5 + (2x + 3)^4 \,.\, 15(3x - 7)^4$$

$$\Rightarrow \quad y' = (2x + 3)^3 (3x - 7)^4 \,(8(3x - 7) + 15(2x + 3))$$

$$\Rightarrow \quad y' = (2x + 3)^3 (3x - 7)^4 \,(24x - 56 + 30x + 45)$$

$$\Rightarrow \quad \boxed{y' = (54x - 11)(2x + 3)^3 (3x - 7)^4}$$

Example 2

Find the gradient of the tangent to $y = \sin x \cos 2x$ at the point $x = \pi/3$.

$$\boxed{\begin{array}{ll} f(x) = \sin x, & g(x) = \cos 2x \\ f'(x) = \cos x, & g'(x) = -2\sin 2x \end{array}}$$

$$y(x) = f(x)\,g(x)$$

$$\therefore \quad y'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\Rightarrow \quad y'(x) = \cos x \,.\, \cos 2x + \sin x \,.\, (-2\sin 2x)$$

$$\Rightarrow \quad y'(x) = \cos x \cos 2x - 2\sin x \sin 2x$$

$$\therefore \quad y'(\pi/3) = \cos(\pi/3)\cos(2\pi/3) - 2\sin(\pi/3)\sin(2\pi/3)$$

$$\Rightarrow \quad y'(\pi/3) = (1/2)(-1/2) - 2(\sqrt{3}/2)(\sqrt{3}/2)$$

$$\Rightarrow \quad y'(\pi/3) = -1/4 - 6/4$$

$$\Rightarrow \quad \boxed{y'(\pi/3) = -7/4}$$

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Quotient Rule

Used to differentiate functions that are divided together.

$$\frac{d}{dx}(f/g) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$

$$D(f/g) = \frac{Df \,.\, g - f \,.\, Dg}{g^2}$$

$$(f/g)' = \frac{f'g - fg'}{g^2}$$

Example 3

Differentiate $y = \dfrac{x^4}{\sin x}$, fully simplifying your answer.

$$f(x) = x^4 \quad , \quad g(x) = \sin x$$
$$f'(x) = 4x^3 \quad , \quad g'(x) = \cos x$$

$$y = \frac{f(x)}{g(x)}$$

$$\therefore \quad y' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

$$\Rightarrow \quad y' = \frac{4x^3 . \sin x - x^4 . \cos x}{(\sin x)^2}$$

$$\Rightarrow \quad \boxed{y' = \frac{x^3(4\sin x - x\cos x)}{\sin^2 x}}$$

Example 4

Show that $y = \dfrac{x^3}{x^2 + 3}$ only has a horizontal tangent at $x = 0$.

$$\boxed{\begin{array}{ll} f(x) = x^3 , & g(x) = x^2 + 3 \\ f'(x) = 3x^2 , & g'(x) = 2x \end{array}}$$

$$y = \frac{f(x)}{g(x)}$$

$$\therefore \quad y' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

$$\Rightarrow \quad y' = \frac{3x^2(x^2 + 3) - x^3(2x)}{(x^2 + 3)^2}$$

$$\Rightarrow \quad y' = \frac{3x^4 + 9x^2 - 2x^4}{(x^2 + 3)^2}$$

$$\Rightarrow \quad y' = \frac{x^4 + 9x^2}{(x^2 + 3)^2}$$

$$\Rightarrow \quad y' = \frac{x^2(x^2 + 9)}{(x^2 + 3)^2}$$

Horizontal tangent means $y' = 0$. So,

$$\frac{x^2(x^2 + 9)}{(x^2 + 3)^2} = 0$$

$$\Rightarrow \quad x^2(x^2 + 9) = 0$$

$$\Rightarrow \quad \boxed{x = 0}$$

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AH Maths - MiA (2nd Edn.)

- pg. 51 Ex. 4.5 All Q.
- pg. 52-3 Ex. 4.6 All Q.
- pg. 53-4 Ex. 4.7 All Q.

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Ex. 4.5

1 Find the derivative of each of these.

a $x^3 \sin x$ b $(x+1)^2 \cos x$ c $(2x+3)^2 \cos 3x$ d $(x+1)^4 (x-1)^3$

e $(3-x)^4 (x+2)^2$ f $x^{-2}(x+4)^3$ g $(x+1)^{-2}(x-1)^2$ h $(x+2)^{-1}(x-1)^{-1}$

2 a If $f(x) = x^3(x+1)^2$, find $f'(1)$.

b If $f(x) = (x-1)^2 \sin x$, find $f'\left(\frac{\pi}{2}\right)$.

3 Differentiate each of these.

a $(2x+1)^2 \cos x$ b $\sin 2x \cos 3x$ c $x^4 \sin 3x$ d $(2x+1)^2(3x-1)^4$

e $(x^2+x) \sin 2x$ f $(x^2-1)(x^3-1)$ g $\sin 2x \sin 3x$ h $x(x^2+3x)^3$

i $x^4(x^2+3x)$ j $\sin x \cos x$ k $\sin(2x+1)\sin(3x+2)$ l $\sin^2 x \cos^2 x$

4 Find the derivative of each of these.

a $(1-5x)^3(1+5x)^2$ b $\sin 3x \cos 5x$

c $x(x+1)^3 \sin 2x$ d $(x+1)^{-2}(2x+1)^2$

5 Find $f'\left(\frac{\pi}{4}\right)$ given that a $f(x) = x\cos^2 x$ b $f(x) = x\cos x^2$

6 Find $f'\left(\frac{\pi}{3}\right)$ given that a $f(x) = (\cos x \sin x)^2$ b $f(x) = \cos x \sin^2 x$

7 If $f(x) = (3x^2+5)(2x^2+3x+1)$ find $f'(0)$.

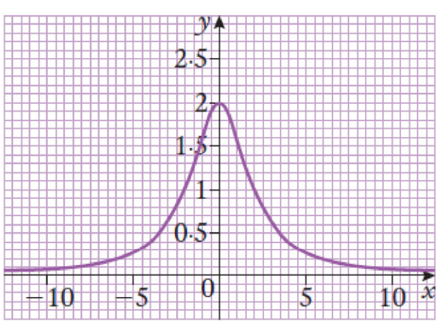
8 $f(x) = \dfrac{x+1}{x-1}$

a By considering the function as $f(x) = (x+1)(x-1)^{-1}$, find $f'(x)$.

b Differentiate i $\dfrac{3x+1}{2x-1}$ ii $\dfrac{3-2x}{x+4}$ iii $\dfrac{\sin x}{\cos x}$

9 In 1630 Pierre De Fermat studied a curve called 'The Witch of Agnesi'.

Its equation is of the form $x^2 y = 4a^2(2a-y)$ where a is a constant.



a In this example $a = 1$. Show that $y = \dfrac{8}{x^2+4}$.

b Find the gradient of the tangent to the curve at the point (2, 1).

Ex. 4.6

1 Differentiate these.

a $\dfrac{x^3}{2x+1}$ b $\dfrac{\cos x}{\sin x}$ c $\dfrac{1-x}{1+x^2}$ d $\dfrac{\sin x}{x}$

2 Find the derivative of

a $\dfrac{x^2}{\sin x}$ b $\dfrac{2x}{\sqrt{x}+5}$ c $\dfrac{x+2}{\sqrt{\sin x}}$ d $\dfrac{\sqrt{x}+1}{2x}$ e $\dfrac{3x+4}{x^{\frac{3}{2}}}$

3 Use the quotient rule to differentiate

a $\dfrac{1}{\sin x}$ b $\dfrac{1}{\cos x}$ c $\dfrac{\sin x}{\cos^2 x}$ d $\dfrac{\sin x}{\cos 2x}$

4 a If $f(x) = \dfrac{x+1}{x^2+2}$ find $f'(0)$. b If $f(x) = \dfrac{x^2}{\sqrt{x}-1}$ find $f'(2)$.

5 Solve $\dfrac{dy}{dx} = 0$ where $y = \dfrac{2x^2+3x-6}{x-2}$.

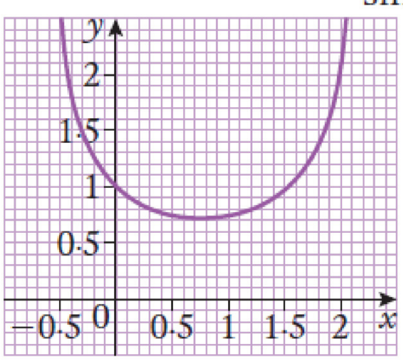
6 a Find $\dfrac{dy}{dx}$ where $y = \dfrac{4}{(x-3)(x+1)}$.

b i Express y in partial fractions.

ii Now find $\dfrac{dy}{dx}$.

c Compare the methods.

7 The graph shows $y = \dfrac{1}{\sin x + \cos x}$; $-0.5 \leqslant x \leqslant 2$.



a Find $\dfrac{dy}{dx}$.

b Show that $\dfrac{dy}{dx} = \dfrac{\sin x - \cos x}{1 + \sin 2x}$.

c Find the gradient of the curve

Ex. 4.7

1 Find the derivative of each of these.

a $\dfrac{x+1}{x^2+3}$ b $\dfrac{1}{\sin 2x}$ c $\sin^2 3x$ d $(x+1)^3(x^3+1)$

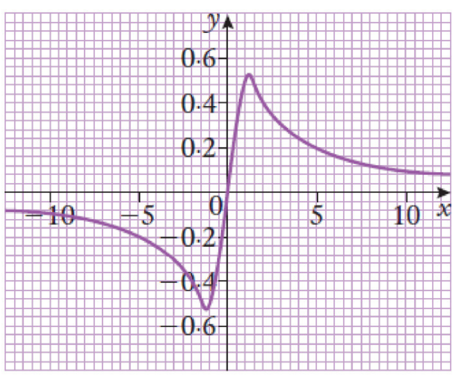
2 If $y = \sin(\sin x)$, find $\dfrac{dy}{dx}$.

3 Find $f'(x)$ when $f(x) = (x+1)^2 \cos 2x$.

4 $y = \dfrac{\cos x}{\cos x + \sin x}$. Show that $\dfrac{dy}{dx} = -\dfrac{1}{1+\sin 2x}$ and hence find the gradient of the tangent at $x = \dfrac{\pi}{4}$.

5 Differentiate $\dfrac{(2x+3)^3}{x^2+3x-1}$.

6 The diagram shows a classic curve, the serpentine curve, whose equation is $x^2y = ax - a^2y$ where a is a constant.



a Make y the subject of the equation.

b Show that $\dfrac{dy}{dx} = \dfrac{a(a^2-x^2)}{(a^2+x^2)^2}$.

c The diagram shows the case where $a = 1$.
What is the gradient at $x = 0$ in this case?

d Let m_1, m_2, m_3 be the gradients of the tangents at $x = \dfrac{a}{2}$, $x = a$ and $x = 2a$ respectively. Show that $m_1 + m_2 + 4m_3 = 0$.

Answers to AH Maths (MiA), pg. 51, Ex. 4.5

1 a $3x^2 \sin x + x^3 \cos x$

b $2(x+1)\cos x - (x+1)^2 \sin x$

c $4(2x+3)\cos 3x - 3(2x+3)^2 \sin 3x$

d $4(x+1)^3(x-1)^3 + 3(x+1)^4(x-1)^2$

e $-4(3-x)^3(x+2)^2 + 2(3-x)^4(x+2)$

f $-2x^{-3}(x+4)^3 + 3x^{-2}(x+4)^2$

g $-2(x+1)^{-3}(x-1)^2 + 2(x+1)^{-2}(x-1)$

h $-(x+2)^{-2}(x-1)^{-1} - (x+2)^{-1}(x-1)^{-2}$

2 a 16 b $\pi - 2$

3 a $4(2x+1)\cos x - (2x+1)^2 \sin x$

b $2\cos 2x \cos 3x - 3\sin 2x \sin 3x$

c $4x^3 \sin 3x + 3x^4 \cos 3x$

d $4(2x+1)(3x-1)^4 + 12(2x+1)^2(3x-1)^3$

e $(2x+1)\sin 2x + 2(x^2+x)\cos 2x$

f $2x(x^3-1) + 3(x^2-1)x^2$

g $2\cos 2x \sin 3x + 3\sin 2x \cos 3x$

h $(x^2+3x)^3 + 3x(2x+3)(x^2+3x)^2$

i $4x^3(x^2+3x) + x^4(2x+3)$

j $\cos^2 x - \sin^2 x = \cos 2x$

k $2\cos(2x+1)\sin(3x+2) + 3\sin(2x+1)\cos(3x+2)$

l $2\sin x \cos^3 x - 2\sin^3 x \cos x$

4 a $-15(1-5x)^2(1+5x)^2 + 10(1-5x)^3(1+5x)$

b $3\cos 3x \cos 5x - 5\sin 3x \sin 5x$

c $(x+1)^2(4x+1)\sin 2x + 2x(x+1)^3\cos 2x$

d $-2(2x+1)^2(x+1)^{-3} + 4(2x+1)(x+1)^{-2}$

5 a $\frac{1}{2} - \frac{\pi}{4}$

b $\cos\left(\frac{\pi^2}{16}\right) - \frac{\pi^2}{8}\sin\frac{\pi^2}{16} \approx 0.102\,047$

6 a $-\frac{\sqrt{3}}{4}$ b $-\frac{\sqrt{3}}{8}$

7 15

8 a $-\frac{2}{(x-1)^2}$

b i $-\frac{5}{(2x-1)^2}$ ii $-\frac{11}{(x+4)^2}$

iii $1 + \tan^2 x$

9 a $x^2 y = 4(2-y) \Rightarrow (x^2+4)y = 8$

b $-\frac{1}{2}$

Answers to AH Maths (MiA), pg. 52-3, Ex. 4.6

1 **a** $\dfrac{4x^3 + 3x^2}{(2x+1)^2}$ **b** $-\dfrac{1}{\sin^2 x}$

c $\dfrac{x^2 - 2x - 1}{(1+x^2)^2}$ **d** $\dfrac{x\cos x - \sin x}{x^2}$

2 **a** $\dfrac{2x\sin x - x^2\cos x}{\sin^2 x}$ **b** $\dfrac{x + 10}{(x+5)^{\frac{3}{2}}}$

c $\dfrac{2\sin x - (x+2)\cos x}{2(\sin x)^{\frac{3}{2}}}$ **d** $\dfrac{-x - 2}{4x^2\sqrt{x+1}}$

e $\dfrac{-3x^{\frac{1}{2}}(x+4)}{2x^3}$

3 **a** $-\dfrac{\cos x}{\sin^2 x}$ **b** $\dfrac{\sin x}{\cos^2 x}$

c $\dfrac{1 + \sin^2 x}{\cos^3 x}$ **d** $\dfrac{\cos 2x\cos x + 2\sin x\sin 2x}{\cos^2 2x}$

4 **a** $\dfrac{1}{2}$ **b** 2

5 $x = 0, 4$

6 **a** $-\dfrac{8(x-1)}{(x-3)^2(x+1)^2}$

b **i** $\dfrac{1}{x-3} - \dfrac{1}{x+1}$ **ii** $-\dfrac{1}{(x-3)^2} + \dfrac{1}{(x+1)^2}$

c The partial fractions method involves more working but simpler derivatives.

7 **a** $\dfrac{\sin x - \cos x}{(\sin x + \cos x)^2}$

b $\dfrac{\sin x - \cos x}{\sin^2 + \cos^2 x + 2\sin x\cos x} = \dfrac{\sin x - \cos x}{1 + \sin 2x}$

c **i** -1 **ii** 1

Answers to AH Maths (MiA), pg. 53-4, Ex. 4.7

1 a $\dfrac{3 - 2x - x^2}{(x^2 + 3)^2}$ b $-\dfrac{2\cos 2x}{\sin^2 2x}$

c $6\sin 3x \cos 3x = 3\sin 6x$

d $3(x + 1)^2(x^3 + 1) + 3x^2(x + 1)^3$

2 $\cos x \cos(\sin x)$

3 $2(x + 1)\cos 2x - 2(x + 1)^2 \sin 2x$

4 $\dfrac{-(\cos x + \sin x)\sin x - \cos x(-\sin x + \cos x)}{(\cos x + \sin x)^2}$

$= \dfrac{-\cos x \sin x - \sin^2 x + \cos x \sin x - \cos^2 x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -\dfrac{1}{2}$

5 $\dfrac{6(x^2 + 3x - 1)(2x + 3)^2 - (2x + 3)^4}{(x^2 + 3x - 1)^2}$

6 a $y = \dfrac{ax}{x^2 + a^2}$

b $\dfrac{(x^2 + a^2)a - ax\,2x}{(x^2 + a^2)^2} = \dfrac{a(x^2 + a^2 - 2x^2)}{(x^2 + a^2)^2}$ hence result.

c 1

d $m_1 = \dfrac{12}{25a}$; $m_2 = 0$; $m_3 = -\dfrac{3}{25a}$ hence result.