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Applications of Calculus - Lesson 2

Optimisation

**LT**
- Solve Optimisation Problems.

**SC**
- Stationary points and values.
- Values of the function at possible endpoints.
Optimisation is finding the 'optimal' solution to a problem.

Examples include: the maximum volume of a cylinder (the radius is the variable); the minimum surface area of metal required to construct a given cuboid.

An optimisation problem for a function $y = f(x)$ normally involves the following steps:

- Make the function $y = f(x)$ from information in the question.
- Find stationary points of $f$ and their nature.
- Work out $y$-values at the stationary points.
- Find $y$-values at any end points.
Example 1

In rectangle PQRS, PQ = 12 cm and QR = 8 cm.

Show that the area of quadrilateral TUVW is given by \( A(x) = 96 - 20x + 2x^2 \) and hence find the minimum area of this quadrilateral.

The area of quadrilateral TUVW is the area of rectangle PQRS minus the areas of the 4 white triangles. Each big triangle has length \((12 - x)\) cm and height \(x\) cm. Each small triangle has length \((8 - x)\) cm and height \(x\) cm. Hence,

\[
A(x) = (12 \times 8) - x(12 - x) - x(8 - x)
\]

\[
A(x) = 96 - 12x + x^2 - 8x + x^2
\]

\[
A(x) = 96 - 20x + 2x^2
\]

\[
A'(x) = -20 + 4x
\]

For a minimum, \(A'(x) = 0\):

\[
-20 + 4x = 0
\]

\[
x = 5
\]
Need to check that $x = 5$ gives a minimum:

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A'(x)$</td>
<td>$-$</td>
<td>0</td>
<td>$+$</td>
</tr>
<tr>
<td>Slope</td>
<td>/</td>
<td></td>
<td>/</td>
</tr>
</tbody>
</table>

$A'(x) = -20 + 4x$

$\therefore A'(4) = -20 + 4(4)$

$\Rightarrow A'(4) = -4 < 0$

$A'(x) = -20 + 4x$

$\therefore A'(4) = -20 + 4(6)$

$\Rightarrow A'(4) = 4 > 0$

$\therefore x = 5\text{ cm}$ gives a minimum value for $A(x)$

The minimum area $A_{\text{MIN}}$ is obtained by putting this $x$-value into the equation for $A(x)$:

$A_{\text{MIN}} = A(5)$

$A_{\text{MIN}} = 96 - 20(5) + 2(5)^2$

$A_{\text{MIN}} = 96 - 100 + 50$

$A_{\text{MIN}} = 46\text{ cm}^2$
Example 2

Plastic storage containers are made from rectangular sheets measuring 2.5 m by 4 m. Two congruent rectangles of width $x$ cm are removed from a sheet.

The remaining plastic is the net of a cuboid; this net is used to create a storage container of height $x$ cm.

Show that the volume, $V$ cm$^3$, of a container is,

$$V(x) = 2x^3 - 650x^2 + 50000x$$

and find the value of $x$ which maximises the volume. Also find the maximum volume.
Let \( L \) and \( B \) be the length and breadth (both in cm), respectively, of the cuboid:

\[
\begin{align*}
2.5 \text{ m} & \\
4 \text{ m} & \\
\end{align*}
\]

After folding, these lengths will be the height \( x \) cm.

The volume of the cuboid is \( V(x) = L \cdot B \cdot H \). The volume must therefore be expressed in terms of the variable \( x \) only. We have, expressing all dimensions in cm,

\[
2x + L = 250
\]

\[
\Rightarrow L = 250 - 2x
\]

Also,

\[
2x + 2B = 400
\]

\[
\Rightarrow B = 200 - x
\]

\[
V(x) = L \cdot B \cdot H
\]

\[
V(x) = (250 - 2x)(200 - x)x
\]

\[
V(x) = (50000 - 650x + 2x^2)x
\]

\[
V(x) = 2x^3 - 650x^2 + 50000x
\]

\[
V(x) = 2x^3 - 650x^2 + 50000x
\]

\[
V'(x) = 6x^2 - 1300x + 50000
\]

For a maximum, \( V'(x) = 0 \):

\[
6x^2 - 1300x + 50000 = 0
\]

This quadratic can be solved by factorising as \((2x - 100)(3x - 500)\) - not obvious!!! - or by using the quadratic formula. The answers for \( x \) are:

\[
x = 50, \quad \frac{500}{3} \quad (\approx 166.67)
\]

Nov 29-09:01
\[ V'(x) = 6x^2 - 1300x + 50000 \]

\[ \therefore V'(40) = 6(40)^2 - 1300(40) + 50000 \]

\[ \Rightarrow V'(40) = 7600 > 0 \]

\[ V'(x) = 6x^2 - 1300x + 50000 \]

\[ \therefore V'(100) = 6(100)^2 - 1300(100) + 50000 \]

\[ \Rightarrow V'(100) = -20000 < 0 \]

\[ V'(x) = 6x^2 - 1300x + 50000 \]

\[ \therefore V'(170) = 6(170)^2 - 1300(170) + 50000 \]

\[ \Rightarrow V'(170) = 2400 > 0 \]

\[ \therefore x = 50 \text{ cm gives a maximum volume} \]

\[ V_{\text{max.}} = V(50) \]

\[ V_{\text{max.}} = 2(50)^3 - 650(50)^2 + 50000(50) \]

\[ V_{\text{max.}} = 250000 - 1625000 + 2500000 \]

\[ V_{\text{max.}} = 1125000 \text{ cm}^3 \]
Example 3

A small baking tin is in the shape of an open-topped, square-based cuboid with volume $500 \text{ cm}^3$. The length of the base is $x \text{ cm}$.

(a) Show that the surface area, $S$, of metal required to make the tin is given by,

$$S(x) = x^2 + \frac{2000}{x}$$

(b) Find the dimensions of the tin which minimise the surface area.

(a) The formula for $S(x)$ only has one variable ($x$); The idea is to use any equations that connect $x$ and $h$ to solve for $h$ (in terms of $x$) and get $S(x)$ in terms of $x$ only.

$$S = x^2 + xh + xh + xh + xh$$
$$S = x^2 + 4xh$$
$$V = x^2h \quad \text{equation connecting } x, h, \text{ and } V$$

$$V = x^2h$$
$$x^2h = 500$$
$$h = \frac{500}{x^2}$$

$$S(x) = x^2 + 4xh$$
$$S(x) = x^2 + 4x \left( \frac{500}{x^2} \right)$$

$$S(x) = x^2 + \frac{2000}{x}$$
(b) \[ S(x) = x^2 + \frac{2000}{x} \]
\[ S(x) = x^2 + 2000x^{-1} \]
\[ \therefore S'(x) = 2x - 2000x^{-2} \]

For a minimum, \( S'(x) = 0 \):
\[ 2x - 2000x^{-2} = 0 \]
\[ x - 1000x^{-2} = 0 \]
\[ x^3 - 1000 = 0 \]
\[ x^3 = 1000 \]
\[ x = 10 \]

Need to check that \( x = 10 \text{ cm} \) gives a minimum:

<table>
<thead>
<tr>
<th>( x )</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S'(x) )</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

\[ S'(x) = 2x - \frac{2000}{x^2} \]
\[ \therefore S'(9) = 2(9) - \frac{2000}{9^2} \]
\[ \Rightarrow S'(9) \approx -6.69 < 0 \]

\[ S'(x) = 2x - \frac{2000}{x^2} \]
\[ \therefore S'(11) = 2(11) - \frac{2000}{11^2} \]
\[ \Rightarrow S'(11) \approx 5.47 > 0 \]

\[ \therefore x = 10 \text{ cm} \text{ gives a minimum value for } S(x) \]

\[ h = \frac{500}{x^2} \]
\[ h = \frac{500}{10^2} \]
\[ h = 5 \text{ cm} \]

Dimensions of minimum surface area:

10 cm by 10 cm by 5 cm

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Example 4

A 2 metre wire is cut $x$ cm from one end and bent to make an equilateral triangle and a square.

The total area of the square and equilateral triangle is given by,

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36} (2 - x)^2$$

Where should the wire be cut for a maximum total area and where should it be cut for a minimum total area?

Find the maximum and minimum total areas to 2 d.p.

We are not asked to derive the equation for the area. So, just differentiate, find stationary points and their nature and state the maximum and minimum values for the area.

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36} (2 - x)^2$$

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18} (2 - x)$$

For maximum or minimum total areas, $A'(x) = 0$:

$$\frac{x}{8} - \frac{\sqrt{3}}{18} (2 - x) = 0$$

After a little manipulation, we find that,

$$x = \frac{8 \sqrt{3}}{9 + 4 \sqrt{3}} \approx 0.869$$

For ease of future reference, we shall denote this $x$-value by $x$. 
\[
\begin{array}{|c|c|c|c|}
\hline
x & 0.7 & 10 & 0.9 \\
\hline
S'(x) & - & 0 & + \\
\hline
\text{Slope} & \downarrow & \text{---} & \downarrow \\
\hline
\end{array}
\]

\[
A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18} (2 - x)
\]

\[
A'(0.7) = \frac{0.7}{8} - \frac{\sqrt{3}}{18} (2 - 0.7)
\]

\[
\Rightarrow A'(0.7) \approx -0.037592 < 0
\]

\[
A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18} (2 - x)
\]

\[
A'(0.9) = \frac{0.9}{8} - \frac{\sqrt{3}}{18} (2 - 0.9)
\]

\[
\Rightarrow A'(0.9) \approx 0.006652 > 0
\]

\[
A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36} (2 - x)^2
\]

\[
A(x_\cdot) = \frac{(x_\cdot)^2}{16} + \frac{\sqrt{3}}{36} (2 - x_\cdot)^2
\]

\[
A(x_\cdot) = \frac{12 + 9\sqrt{3}}{(9 + 4\sqrt{3})^2}
\]

\[
A(x_\cdot) \approx 0.1087 \ldots
\]

At the endpoints,

\[
A(0) = \frac{(0)^2}{16} + \frac{\sqrt{3}}{36} (2 - 0)^2
\]

\[
A(0) = \frac{\sqrt{3}}{9}
\]

\[
A(0) \approx 0.1924 \ldots
\]

\[
A(2) = \frac{(2)^2}{16} + \frac{\sqrt{3}}{36} (2 - 2)^2
\]

\[
A(2) = 0.25
\]

Wire must be cut at 2 cm from one end to give a maximum area of 0.25 cm\(^2\). Wire must be cut at x\(\cdot\) (\(\approx 0.87\) cm) from that same end to give a minimum area of 0.11 cm\(^2\) (to 2 d. p.).
CfE Higher Maths

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