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Applications of Calculus - Lesson 2

Optimisation

LI

- Solve Optimisation Problems.

SC

- Stationary points and values.
- Values of the function at possible endpoints.

Optimisation is finding the 'optimal' solution to a problem.

Examples include : the maximum volume of a cylinder (the radius is the variable); the minimum surface area of metal required to construct a given cuboid.

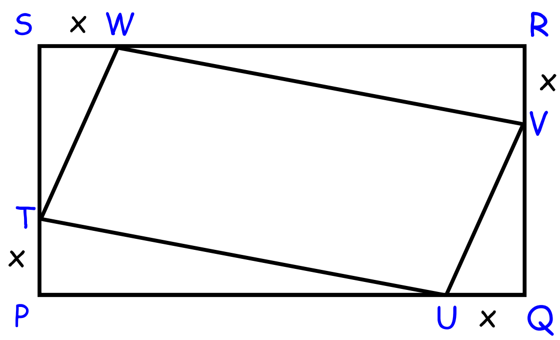
An optimisation problem for a function $y = f(x)$ normally involves the following steps :

- Make the function $y = f(x)$ from information in the question.
- Find stationary points of f and their nature.
- Work out y - values at the stationary points.
- Find y - values at any end points.

Example 1

In rectangle PQRS, PQ $=$ 12 cm and QR $=$ 8 cm.

Show that the area of quadrilateral TUVW is given by $A(x) = 96 - 20x + 2x^2$ and hence find the minimum area of this quadrilateral.



The area of quadrilateral TUVW is the area of rectangle PQRS minus the areas of the 4 white triangles. Each big triangle has length $(12 - x)$ cm and height x cm. Each small triangle has length $(8 - x)$ cm and height x cm. Hence,

$$A(x) = (12 \times 8) - x(12 - x) - x(8 - x)$$

$$A(x) = 96 - 12x + x^2 - 8x + x^2$$

$$\boxed{A(x) = 96 - 20x + 2x^2}$$

$$A(x) = 96 - 20x + 2x^2$$

$$A'(x) = -20 + 4x$$

For a minimum, $A'(x) = 0$:

$$-20 + 4x = 0$$

$$\underline{x = 5}$$

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Need to check that $x = 5$ gives a minimum :

x	$\xrightarrow{4}$	5	$\xrightarrow{6}$
$A'(x)$	$-$	0	$+$
Slope	$\searrow$	—	$\nearrow$

$$A'(x) = -20 + 4x$$

$$\therefore \quad A'(4) = -20 + 4(4)$$

$$\Rightarrow \quad A'(4) = -4 < 0$$

$$A'(x) = -20 + 4x$$

$$\therefore \quad A'(4) = -20 + 4(6)$$

$$\Rightarrow \quad A'(4) = 4 > 0$$

$\therefore$ $x = 5$ cm gives a minimum value for $A(x)$

The minimum area $A_{MIN.}$ is obtained by putting this x - value into the equation for $A(x)$:

$$A_{MIN.} = A(5)$$

$$A_{MIN.} = 96 - 20(5) + 2(5)^2$$

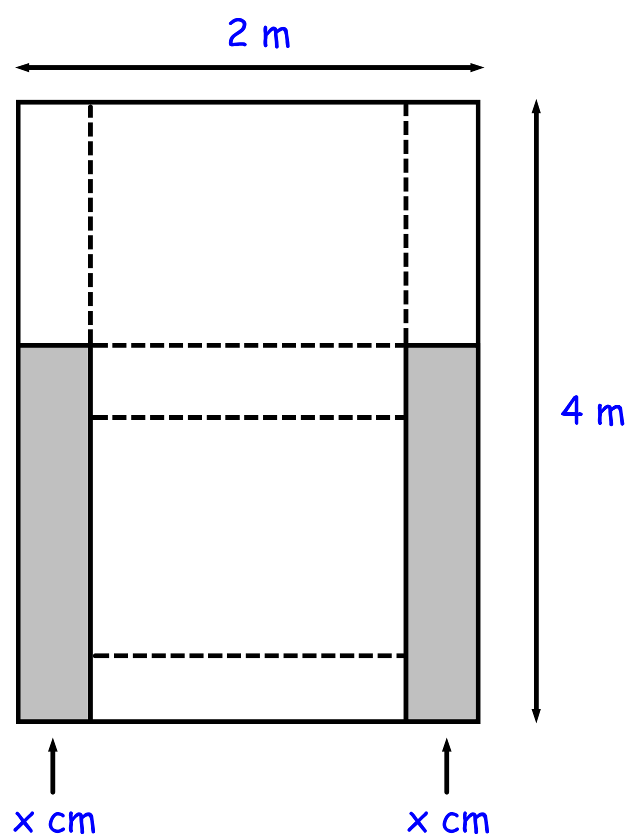
$$A_{MIN.} = 96 - 100 + 50$$

$$\boxed{A_{MIN.} = 46 \text{ cm}^2}$$

Example 2

Plastic storage containers are made from rectangular sheets measuring 2.5 m by 4 m. Two congruent rectangles of width x cm are removed from a sheet.

The remaining plastic is the net of a cuboid; this net is used to create a storage container of height x cm.



Show that the volume, V cm^3, of a container is,

$$V(x) = 2x^3 - 650x^2 + 50\,000x$$

and find the value of x which maximises the volume. Also find the maximum volume.

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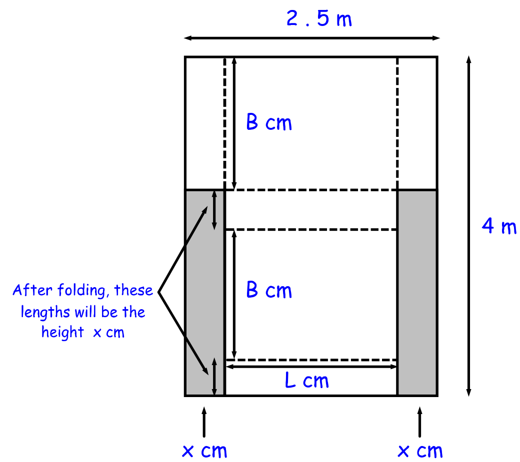
Let L and B be the length and breadth (both in cm), respectively, of the cuboid :



The volume of the cuboid is $V(x) = LBH$. The volume must therefore be expressed in terms of the variable x only. We have, expressing all dimensions in cm,

$$2x + L = 250$$
$$\Rightarrow \quad \underline{L = 250 - 2x}$$

Also,

$$2x + 2B = 400$$
$$\Rightarrow \quad \underline{B = 200 - x}$$

$$V(x) = LBH$$
$$V(x) = (250 - 2x)(200 - x)x$$
$$V(x) = (50\,000 - 650x + 2x^2)x$$
$$\boxed{V(x) = 2x^3 - 650x^2 + 50\,000x}$$

$$V(x) = 2x^3 - 650x^2 + 50\,000x$$
$$V'(x) = 6x^2 - 1\,300x + 50\,000$$

For a maximum, $V'(x) = 0$:

$$6x^2 - 1\,300x + 50\,000 = 0$$

This quadratic can be solved by factorising as $(2x - 100)(3x - 500)$ - not obvious !!! - or by using the quadratic formula. The answers for x are :

$$\underline{x = 50, \ \frac{500}{3} \ (\approx 166.67)}$$

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x	$\xrightarrow{40}$	50	$\xrightarrow{100}$	$\frac{500}{3}$	$\xrightarrow{170}$
$f'(x)$	$+$	0	$-$	0	$+$
Slope	/	—	\	—	/

$$V'(x) = 6x^2 - 1\,300x + 50\,000$$

$$\therefore \quad V'(40) = 6(40)^2 - 1\,300(40) + 50\,000$$

$$\Rightarrow \quad \underline{V'(40) = 7\,600 > 0}$$

$$V'(x) = 6x^2 - 1\,300x + 50\,000$$

$$\therefore \quad V'(100) = 6(100)^2 - 1\,300(100) + 50\,000$$

$$\Rightarrow \quad \underline{V'(100) = -20\,000 < 0}$$

$$V'(x) = 6x^2 - 1\,300x + 50\,000$$

$$\therefore \quad V'(170) = 6(170)^2 - 1\,300(170) + 50\,000$$

$$\Rightarrow \quad \underline{V'(170) = 2\,400 > 0}$$

$$\therefore \quad \underline{x = 50 \text{ cm gives a maximum volume}}$$

$$V_{MAX.} = V(50)$$

$$V_{MAX.} = 2(50)^3 - 650(50)^2 + 50\,000(50)$$

$$V_{MAX.} = 250\,000 - 1\,625\,000 + 2\,500\,000$$

$$\boxed{V_{MAX.} = 1\,125\,000 \text{ cm}^3}$$

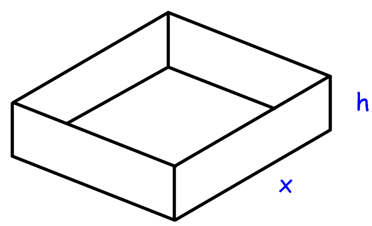
Example 3

A small baking tin is in the shape of an open-topped, square-based cuboid with volume 500 cm ^{3}. The length of the base is x cm.

(a) Show that the surface area, S, of metal required to make the tin is given by,

$$S(x) = x^2 + \frac{2\,000}{x}$$

(b) Find the dimensions of the tin which minimise the surface area.



(a) The formula for $S(x)$ only has one variable (x); The idea is to use any equations that connect x and h to solve for h (in terms of x) and get $S(x)$ in terms of x only.

$$S = x^2 + xh + xh + xh + xh$$

$$S = x^2 + 4xh$$

$$V = x^2 h \longleftarrow$$ equation connecting x, h, and V

$$V = x^2 h$$

$$x^2 h = 500$$

$$h = \frac{500}{x^2}$$

$$S(x) = x^2 + 4xh$$

$$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right)$$

$$\boxed{S(x) = x^2 + \frac{2\,000}{x}}$$

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(b)

$$S(x) = x^2 + \frac{2\,000}{x}$$

$$S(x) = x^2 + 2\,000\,x^{-1}$$

$$\therefore \quad S'(x) = 2x - 2\,000\,x^{-2}$$

For a minimum, $S'(x) = 0$:

$$2x - 2\,000\,x^{-2} = 0$$

$$x - 1\,000\,x^{-2} = 0$$

$$x^3 - 1\,000 = 0$$

$$x^3 = 1\,000$$

$$\underline{x = 10}$$

Need to check that $x = 10$ cm gives a minimum :

x	$\xrightarrow{9}$	10	$\xrightarrow{11}$
$S'(x)$	$-$	0	$+$
Slope	$\searrow$	$—$	$\nearrow$

$$S'(x) = 2x - \frac{2\,000}{x^2}$$

$$\therefore \quad S'(9) = 2(9) - \frac{2\,000}{9^2}$$

$$\Rightarrow \quad \underline{S'(9) \approx -6.69 < 0}$$

$$S'(x) = 2x - \frac{2\,000}{x^2}$$

$$\therefore \quad S'(11) = 2(11) - \frac{2\,000}{11^2}$$

$$\Rightarrow \quad \underline{S'(11) \approx 5.47 > 0}$$

$\therefore$ $\boxed{x = 10 \text{ cm gives a minimum value for } S(x)}$

$$h = \frac{500}{x^2}$$

$$h = \frac{500}{10^2}$$

$$\underline{h = 5 \text{ cm}}$$

Dimensions of minimum surface area :

10 cm by 10 cm by 5 cm

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Example 4

A 2 metre wire is cut x cm from one end and bent to make an equilateral triangle and a square.

The total area of the square and equilateral triangle is given by,

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(2 - x)^2$$

Where should the wire be cut for a maximum total area and where should it be cut for a minimum total area ?

Find the maximum and minimum total areas to 2 d.p. .

We are not asked to derive the equation for the area. So, just differentiate, find stationary points and their nature and state the maximum and minimum values for the area.

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(2 - x)^2$$

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(2 - x)$$

For maximum or minimum total areas, $A'(x) = 0$:

$$\frac{x}{8} - \frac{\sqrt{3}}{18}(2 - x) = 0$$

After a little manipulation, we find that, :

$$x = \frac{8\sqrt{3}}{9 + 4\sqrt{3}} \approx 0.869\ldots$$

For ease of future reference, we shall denote this x - value by x_c.

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x	$\xrightarrow{0.7}$	10	$\xrightarrow{0.9}$
$S'(x)$	$-$	0	$+$
Slope	╲	—	╱

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(2 - x)$$

$$\therefore \quad A'(0.7) = \frac{0.7}{8} - \frac{\sqrt{3}}{18}(2 - 0.7)$$

$$\Rightarrow \quad \underline{A'(0.7) \approx -0.037\,592 < 0}$$

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(2 - x)$$

$$\therefore \quad A'(0.9) = \frac{0.9}{8} - \frac{\sqrt{3}}{18}(2 - 0.9)$$

$$\Rightarrow \quad \underline{A'(0.9) \approx 0.006\,652 > 0}$$

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(2 - x)^2$$

$$A(x_c) = \frac{(x_c)^2}{16} + \frac{\sqrt{3}}{36}(2 - x_c)^2$$

$$A(x_c) = \frac{12 + 9\sqrt{3}}{\left(9 + 4\sqrt{3}\right)^2}$$

$$\underline{A(x_c) \approx 0.108\,7\ldots}$$

At the endpoints,

$$A(0) = \frac{(0)^2}{16} + \frac{\sqrt{3}}{36}(2 - 0)^2$$

$$A(0) = \frac{\sqrt{3}}{9}$$

$$\underline{A(0) \approx 0.192\,4\ldots}$$

$$A(2) = \frac{(2)^2}{16} + \frac{\sqrt{3}}{36}(2 - 2)^2$$

$$\underline{A(2) = 0.25}$$

Wire must be cut at 2 cm from one end to give a maximum area of $0.25\ \text{cm}^2$. Wire must be cut at x_c (≈ 0.87) cm from that same end to give a minimum area of $0.11\ \text{cm}^2$ (to 2 d. p.).

CfE Higher Maths

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