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Unit 2 : Proof by Mathematical Induction - Lesson 2

Mathematical Induction 2 (Divisibility)

LI

- Use Proof by Mathematical Induction to solve problems involving divisibility.

SC

- Algebra.

Jan 10-12:11

Proof by Mathematical Induction

The Principle of Mathematical Induction (PMI) states that, to prove a statement ($P(n)$) about an infinite set of natural numbers :

- Prove the Base Case : $P(n_0)$ is true.
- Prove the Inductive Step :
  $P(k)$ true $\Rightarrow$ $P(k+1)$ true.

Then $P(n)$ is true $\forall\, n \geq n_0$.

Usually, $n_0 = 1$; then we would state the conclusion as '$P(n)$ is true $\forall\, n \in \mathbb{N}$'.

Divisibility

If a natural number A is divisible by a natural number B, this is written $B \mid A$ ('B divides A' or 'A is divisible by B'). Then there exists a natural number w such that $A = B w$.

So, for $A, B \in \mathbb{N}$,

$$B \mid A \text{ is equivalent to } \exists\, w \in \mathbb{N} \text{ such that } A = B w.$$

Nov 26-11:16

Example 1

Prove by induction that $7^n - 1$ is divisible by 6 $\forall n \in \mathbb{N}$.

Proving that $7^n - 1$ is divisible by 6 is equivalent to proving that $\exists w \in \mathbb{N}$ such that $7^n - 1 = 6w$.

$$P(n): 7^n - 1 = 6w \text{ (for some } w \in \mathbb{N})$$

Base Case

$$\boxed{\text{As } 7^1 - 1 = 6 = 6 \,.\, 1, P(1) \text{ is true}}$$

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$7^k - 1 = 6r \text{ (for some } r \in \mathbb{N}) \longleftarrow$$

Inductive Hypothesis

RTP statement

$$7^{k+1} - 1 = 6u \text{ (for some } u \in \mathbb{N})$$

$$7^{k+1} - 1 = 7^k \,.\, 7 - 1$$

$$\Rightarrow \quad 7^{k+1} - 1 = (6r + 1) \,.\, 7 - 1$$

$$\Rightarrow \quad 7^{k+1} - 1 = 6 \,.\, 7r + 7 - 1$$

$$\Rightarrow \quad 7^{k+1} - 1 = 6 \,.\, 7r + 6$$

$$\Rightarrow \quad 7^{k+1} - 1 = 6(7r + 1)$$

$$\Rightarrow \quad 7^{k+1} - 1 = 6u$$

As $r \in \mathbb{N}$, $u = 7r + 1 \in \mathbb{N}$.

$$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true}}$$

'P (1) true' and 'P (k) true ⇒ P (k + 1) true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

Oct 2-23:10

Example 2

Prove by induction that $8^n - 3^n$ is divisible by $5 \ \forall n \in \mathbb{N}$.

$$P(n): 8^n - 3^n = 5w \text{ (for some } w \in \mathbb{N})$$

Base Case

$\boxed{\text{As } 8^1 - 3^1 = 5 = 5.1, P(1) \text{ is true}}$

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$8^k - 3^k = 5r \text{ (for some } r \in \mathbb{N}) \longleftarrow$$ Inductive Hypothesis

RTP statement

$$8^{k+1} - 3^{k+1} = 5u \text{ (for some } u \in \mathbb{N})$$

$$8^{k+1} - 3^{k+1} = 8^k . 8 - 3^{k+1}$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = (5r + 3^k) . 8 - 3^{k+1}$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = 5 . 8r + 8 . 3^k - 3^{k+1}$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = 5 . 8r + 8 . 3^k - 3 . 3^k$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = 5 . 8r + 5 . 3^k$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = 5(8r + 3^k)$$

$$\Rightarrow \quad 8^{k+1} - 3^{k+1} = 5u$$

As $r, k \in \mathbb{N}, u = 8r + 3^k \in \mathbb{N}$.

$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true}}$

'P (1) true' and 'P (k) true $\Rightarrow$ P (k + 1) true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

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Example 3

Prove by induction that $2^{3n+2} + 5^{n+1}$ is divisible by $3 \; \forall \, n \in \mathbb{N}$.

$$P(n): \; 2^{3n+2} + 5^{n+1} = 3w \; (\text{for some } w \in \mathbb{N})$$

Base Case

As $2^{3+2} + 5^{1+1} = 32 + 25 = 57 = 3 \, . \, 19$, $P(1)$ is true

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$2^{3k+2} + 5^{k+1} = 3r \; (\text{for some } r \in \mathbb{N}) \longleftarrow \text{Inductive Hypothesis}$$

RTP statement

$$2^{3(k+1)+2} + 5^{k+2} = 3u \; (\text{for some } u \in \mathbb{N})$$

$$2^{3(k+1)+2} + 5^{k+2} = 2^{3k+3+2} + 5^{k+2}$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = 2^{3k+2} \, . \, 2^{3} + 5^{k+2}$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = (3r - 5^{k+1}) \, . \, 8 + 5^{k+2}$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = 3 \, . \, 8r - 8 \, . \, 5^{k+1} + 5 \, . \, 5^{k+1}$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = 3 \, . \, 8r - 3 \, . \, 5^{k+1}$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = 3(8r - 5^{k+1})$$

$$\Rightarrow \; 2^{3(k+1)+2} + 5^{k+2} = 3u$$

As $r, k \in \mathbb{N}$, $u = 8r - 5^{k+1} \in \mathbb{N}$ (as $3r > 5^{k+1}$).

Hence, $P(k)$ true $\Rightarrow$ $P(k+1)$ true

'$P(1)$ true' and '$P(k)$ true $\Rightarrow$ $P(k+1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall \, n \in \mathbb{N}$

Questions (and 'Answers' !)

Prove by mathematical induction that $\forall n \in \mathbb{N}$:

1) $5^{2n} - 1$ is divisible by 24.

2) $n^3 + 5n + 6$ is divisible by 3.

3) $4^{3n} + 8$ is divisible by 9.

4) $4^{n+1} + 5^{2n-1}$ is divisible by 21.

5) $5^{2n} - 4^{2n}$ is divisible by 9.

6) $7^{2n} + 16n - 1$ is divisible by 64.