21 / 12 / 16

Recurrence Relations - Lesson 2

Limits of Recurrence Relations

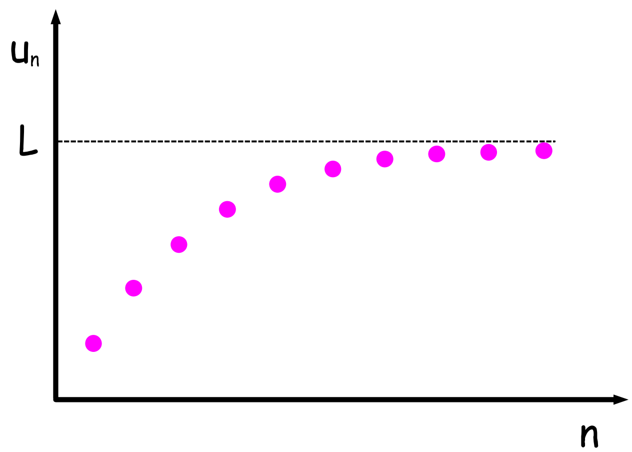
LI

- Know what a limit of a recurrence relations is.
- Find a limit of a recurrence relation (calc. and non-calc.).
- Solve limit problems in context.

SC

- Use Limit Formula.

Given a linear recurrence relation $u_{n+1} = a u_n + b$, if n is very large, the terms u_n and u_{n+1} can be very close together.
If the terms of the recurrence relation approach some number L, then, as $n \longrightarrow \infty$, $u_n \longrightarrow L$ and $u_{n+1} \longrightarrow L$:



As $n \longrightarrow \infty$, $u_n \longrightarrow L$ and $u_{n+1} \longrightarrow L$. Using this information in the recurrence relation $u_{n+1} = a u_n + b$,

$$L = aL + b$$

$$L(1 - a) = b$$

$$L = \frac{b}{1-a} \qquad (a \neq 1)$$

Some recurrence relations don't have a limiting value; this will happen whenever $a \geq 1$ or $a \leq -1$.

The recurrence relation $u_{n+1} = a u_n + b$ has a limit(ing value) provided that $-1 < a < 1$; the limit L is then given by,

$$L = \frac{b}{1-a} \qquad (-1 < a < 1)$$

Dec 21-08:14

Example 1 (Non-Calc)

Find the limit of the recurrence relation $u_{n+1} = 0.4u_n + 18$.

$$u_{n+1} = 0.4u_n + 18$$

$$u_{n+1} = a u_n + b$$

$$a = 0.4 = \frac{4}{10} = \frac{2}{5}$$

$$b = 18$$

$$L = \frac{b}{1 - a}$$

$$L = \frac{18}{1 - 2/5}$$

$$L = \frac{18}{3/5}$$

$$\boxed{L = 30}$$

Example 2 (Non-Calc)

If the limit of the recurrence relation $u_{n+1} = p\,u_n + 6$ is 12, find the value of p.

$$u_{n+1} = p\,u_n + 6$$

$$L = \frac{b}{1 - a}$$

$$12 = \frac{6}{1 - p}$$

$$1 - p = 1/2$$

$$\boxed{p = 1/2}$$

Example 3 (Non-Calc)

Find the range of values of k for the recurrence relation $u_{n+1} = 3ku_n + 12$ to have a limit.

For the recurrence relation $u_{n+1} = au_n + b$ to have a limit, we must have $-1 < a < 1$. So,

$$-1 < 3k < 1$$

$$\boxed{-1/3 < k < 1/3}$$

Mar 3-16:58

Example 4 (Calc)

A chemical factory wants to dump 200 kg of waste into a sea loch every week. It is estimated that the tide will remove 65 % of this waste each week. Environmental groups claim that more than 310 kg of waste in the loch will be harmful to the local seal colony.

In the long-term, is it safe for the factory to dump the waste ?

The phrase 'long-term' means 'as $n \longrightarrow \infty$'. We first need to set up the recurrence relation, justify that a limit exists and then find the limit.

Let u_n be the amount of waste (in kg) in the loch at the end of the n^{th} week. Then, as 35 % is still left and 200 kg is added each week, we have,

$$u_{n+1} = 0.35\,u_n + 200$$

As $-1 < 0.35 < 1$, a limit exists for this recurrence relation.

$$L = \frac{b}{1 - a}$$

$$L = \frac{200}{1 - 0.35}$$

$$L = \frac{200}{0.65}$$

$$\underline{L = 307.69\ldots}$$

As the long-term waste (307 . 7 kg) is less than 310 kg, it is safe for the factory to dump the waste.

CfE Higher Maths

pg. 321 - 2 Ex. 15D All Q