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Recurrence Relations - Lesson 2

Limits of Recurrence Relations

**LI**
- Know what a limit of a recurrence relations is.
- Find a limit of a recurrence relation (calc. and non-calc.).
- Solve limit problems in context.

**SC**
- Use Limit Formula.
Given a linear recurrence relation \( u_{n+1} = au_n + b \), if \( n \) is very large, the terms \( u_n \) and \( u_{n+1} \) can be very close together. If the terms of the recurrence relation approach some number \( L \), then, as \( n \to \infty, u_n \to L \) and \( u_{n+1} \to L \):

As \( n \to \infty, u_n \to L \) and \( u_{n+1} \to L \). Using this information in the recurrence relation \( u_{n+1} = au_n + b \),

\[
L = aL + b
\]

\[
L(1 - a) = b
\]

\[
L = \frac{b}{1 - a} \quad (a \neq 1)
\]

Some recurrence relations don't have a limiting value; this will happen whenever \( a \geq 1 \) or \( a \leq -1 \).

The recurrence relation \( u_{n+1} = au_n + b \) has a limit(ing value) provided that \(-1 < a < 1\); the limit \( L \) is then given by,

\[
L = \frac{b}{1 - a} \quad (-1 < a < 1)
\]
**Example 1** (Non-Calc)

Find the limit of the recurrence relation 
\[ u_{n+1} = 0.4 \, u_n + 18. \]

\[ u_{n+1} = 0.4 \, u_n + 18 \]
\[ u_{n+1} = a \, u_n + b \]

\[ a = 0.4 = \frac{4}{10} = \frac{2}{5} \]
\[ b = 18 \]

\[ L = \frac{b}{1 - a} \]
\[ L = \frac{18}{1 - \frac{2}{5}} \]
\[ L = \frac{18}{3/5} \]

**L = 30**
Example 2 (Non-Calc)

If the limit of the recurrence relation
\( u_{n+1} = p \cdot u_n + 6 \) is 12, find the value of \( p \).

\[
\begin{align*}
    u_{n+1} &= p \cdot u_n + 6 \\
    L &= \frac{b}{1 - a} \\
    12 &= \frac{6}{1 - p} \\
    1 - p &= 1/2 \\
    p &= 1/2
\end{align*}
\]
Example 3 (Non-Calc)

Find the range of values of $k$ for the recurrence relation $u_{n+1} = 3k u_n + 12$ to have a limit.

For the recurrence relation $u_{n+1} = au_n + b$ to have a limit, we must have $-1 < a < 1$. So,

$$-1 < 3k < 1$$

$$-1/3 < k < 1/3$$
Example 4 (Calc)

A chemical factory wants to dump 200 kg of waste into a sea loch every week. It is estimated that the tide will remove 65% of this waste each week. Environmental groups claim that more than 310 kg of waste in the loch will be harmful to the local seal colony.

In the long-term, is it safe for the factory to dump the waste?

The phrase 'long-term' means 'as $n \to \infty$'. We first need to set up the recurrence relation, justify that a limit exists and then find the limit.

Let $u_n$ be the amount of waste (in kg) in the loch at the end of the $n^{th}$ week. Then, as 35% is still left and 200 kg is added each week, we have,

$$u_{n+1} = 0.35 u_n + 200$$

As $-1 < 0.35 < 1$, a limit exists for this recurrence relation.

$$L = \frac{b}{1 - a}$$

$$L = \frac{200}{1 - 0.35}$$

$$L = \frac{200}{0.65}$$

$$L = 307.69 \ldots$$

As the long-term waste (307.7 kg) is less than 310 kg, it is safe for the factory to dump the waste.
CfE Higher Maths

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