

29 / 8 / 17

*Unit 1 : Integral Calculus - Lesson 2*

## Integration by Substitution

LI

- Integrate using a change of variable.

SC

- Substitution technique.

## Differentials

A differential is a small change in the values of a function  $y = f(x)$ , specified by,

$$f'(x) = \frac{dy}{dx} \Rightarrow dy = f'(x) dx$$

Differentials

**Warning for future maths study :**

$\frac{dy}{dx}$  is not to be viewed as a fraction

### Examples

- $y = x^3 \Rightarrow dy = 3x^2 dx$
- $u = \cos x \Rightarrow du = -\sin x dx$

Integration by Substitution is an integration technique that involves using a new letter (aka substitution variable) to simplify an integral

Differentials are used to ultimately change the integral into a standard integral

If part of the integrand is the derivative of another part of the integrand, then integration by substitution is the strategy to opt for

For some integrals, the substitution is obvious; for more difficult integrals, the substitution is given and some rearranging of the integrand may be necessary before a standard integral is revealed

For definite integrals, the integration limits must be changed from the original variable (normally  $x$ ) to the substitution variable (normally  $u$ )

Example 1

Integrate  $y = 4x^3(x^4 + 7)^6$ .

$$\text{Let } I = \int 4x^3(x^4 + 7)^6 dx.$$

Notice that  $4x^3$  is the derivative of  $x^4 + 7$ . We thus let our new integration variable (u) be equal to  $x^4 + 7$ .

$$\begin{aligned} u &= x^4 + 7 \\ \Rightarrow du &= 4x^3 dx \end{aligned}$$

$$I = \int 4x^3(x^4 + 7)^6 dx$$

$$\therefore I = \int u^6 du$$

$$\Rightarrow I = \frac{1}{7} u^7 + C$$

$$\Rightarrow I = \frac{1}{7} (x^4 + 7)^7 + C$$

Example 2

Integrate  $y = (6x^2 + 8x + 18)e^{(x^3 + 2x^2 + 9x)}$ .

Let  $I = \int (6x^2 + 8x + 18)e^{(x^3 + 2x^2 + 9x)} dx$ .

$$\begin{aligned} u &= x^3 + 2x^2 + 9x \\ \Rightarrow du &= (3x^2 + 4x + 9) dx \\ \Rightarrow 2du &= (6x^2 + 8x + 18) dx \end{aligned}$$

$$I = \int (6x^2 + 8x + 18)e^{(x^3 + 2x^2 + 9x)} dx$$

$$\therefore I = 2 \int e^u du$$

$$\Rightarrow I = 2e^u + C$$

$$\Rightarrow I = 2e^{(x^3 + 2x^2 + 9x)} + C$$

Example 3

Integrate  $y = \operatorname{cosec}^2(3x) e^{\cot(3x)}$ .

Let  $I = \int \operatorname{cosec}^2(3x) e^{\cot(3x)} dx$ .

$$\begin{aligned} u &= \cot(3x) \\ \Rightarrow du &= -3 \operatorname{cosec}^2(3x) dx \\ \Rightarrow -\frac{1}{3} du &= \operatorname{cosec}^2(3x) dx \end{aligned}$$

$$I = \int \operatorname{cosec}^2(3x) e^{\cot(3x)} dx$$

$$\therefore I = -\frac{1}{3} \int e^u du$$

$$\Rightarrow I = -\frac{1}{3} e^u + C$$

$$\Rightarrow I = -\frac{1}{3} e^{\cot(3x)} + C$$

Example 4

$$\text{Integrate } y = \frac{\sec^2(\ln x)}{x} .$$

$$\text{Let } I = \int \frac{\sec^2(\ln x)}{x} dx .$$

$$\begin{aligned} u &= \ln x \\ \Rightarrow du &= \frac{1}{x} dx \end{aligned}$$

$$I = \int \frac{\sec^2(\ln x)}{x} dx$$

$$\therefore I = \int \sec^2 u du$$

$$\Rightarrow I = \tan u + C$$

$$\Rightarrow I = \tan(\ln x) + C$$

Example 5

Use the substitution  $x = 4 \sin u$  to evaluate the integral,

$$\int \sqrt{16 - x^2} dx.$$

$$\text{Let } I = \int \sqrt{16 - x^2} dx.$$

$$\boxed{\begin{aligned} x &= 4 \sin u \\ \Rightarrow dx &= 4 \cos u du \end{aligned}}$$

$$I = \int \sqrt{16 - x^2} dx$$

$$\therefore I = \int \sqrt{16 - (4 \sin u)^2} \cdot 4 \cos u du$$

$$\Rightarrow I = \int \sqrt{16 - 16 \sin^2 u} \cdot 4 \cos u du$$

$$\Rightarrow I = \int \sqrt{16(1 - \sin^2 u)} \cdot 4 \cos u du$$

$$\Rightarrow I = \int 4 \cos u \cdot 4 \cos u du$$

$$\Rightarrow I = 16 \int \cos^2 u du$$

The trigonometric identity  $\cos 2u = 2 \cos^2 u - 1$  can be rearranged to solve for  $\cos^2 u$  giving,

$$I = 16 \int (1/2)(1 + \cos 2u) du$$

$$\Rightarrow I = 8 \int (1 + \cos 2u) du$$

$$\Rightarrow I = 8u + 4 \sin 2u + C$$

$$\Rightarrow I = 8u + 4(2 \sin u \cos u) + C$$

Using  $x = 4 \sin u$ , we have, using the identity  $\sin^2 u + \cos^2 u = 1$ ,  $\cos u = (1/4) \sqrt{16 - x^2}$ .

Hence,

$$I = 8 \sin^{-1}(x/4) + \frac{x \sqrt{16 - x^2}}{2} + C$$

## Two Useful Integrals

Differentiating  $\sin^{-1}(x/a)$  gives (a is constant),

$$\begin{aligned}\frac{d}{dx} \sin^{-1}(x/a) &= \frac{\frac{1}{a}}{\sqrt{1 - (x/a)^2}} \\ &= \frac{\frac{1}{a}}{\frac{1}{a} \sqrt{a^2 - x^2}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}\end{aligned}$$

Hence,

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(x/a) + C$$

It can similarly be shown that,

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(x/a) + C$$

The above two integrals can also be obtained by making the substitution  $x = a u$ .

Example 6

Integrate  $\frac{1}{\sqrt{1 - 4x^2}}$ .

$$\text{Let } I = \int \frac{1}{\sqrt{1 - 4x^2}} dx.$$

$$I = \int \frac{1}{\sqrt{1 - 4x^2}} dx$$

$$\therefore I = \int \frac{1}{\sqrt{4((1/4) - x^2)}} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{(1/2)^2 - x^2}} dx$$

$$\Rightarrow I = \frac{1}{2} \sin^{-1}(x/(1/2)) + C$$

$$\Rightarrow I = \frac{1}{2} \sin^{-1}(2x) + C$$

This integral can also be obtained by making the substitution  $x = (1/4)u$ .

Example 7

Integrate  $\frac{1}{1 + 9x^2}$ .

$$\text{Let } I = \int \frac{1}{1 + 9x^2} dx.$$

$$I = \int \frac{1}{1 + 9x^2} dx$$

$$\therefore I = \int \frac{1}{9(1/9 + x^2)} dx$$

$$\Rightarrow I = \frac{1}{9} \int \frac{1}{(1/3)^2 + x^2} dx$$

$$\Rightarrow I = \frac{1}{9} (1/(1/3)) \tan^{-1}(x/(1/3)) + C$$

$$\Rightarrow I = \frac{1}{3} \tan^{-1}(3x) + C$$

This integral can also be obtained by making the substitution  $x = (1/3)u$ .

Example 8

Show that  $\int_1^2 2x e^{x^2} dx = e(e^3 - 1)$ .

Let  $I = \int_1^2 2x e^{x^2} dx$ .

$$\begin{aligned} u &= x^2 \\ \Rightarrow du &= 2x dx \\ x = 1 &\Rightarrow u = 1^2 \Rightarrow u = 1 \\ x = 2 &\Rightarrow u = 2^2 \Rightarrow u = 4 \end{aligned}$$

$$I = \int_1^2 2x e^{x^2} dx$$

$$\therefore I = \int_1^4 e^u du$$

$$\Rightarrow I = [e^u]_1^4$$

$$\Rightarrow I = e^4 - e^1$$

$$\Rightarrow I = e^4 - e$$

$$\Rightarrow I = e(e^3 - 1)$$

Example 9

Express  $\int_1^2 \frac{2x + 3}{x^2 + 3x} dx$  in the form  $\ln(a/b)$ ,

where the fraction is fully simplified, stating the values of  $a$  and  $b$ .

$$\text{Let } I = \int_1^2 \frac{2x + 3}{x^2 + 3x} dx.$$

$$u = x^2 + 3x$$

$$\Rightarrow du = (2x + 3) dx$$

$$x = 1 \Rightarrow u = 1^2 + 3(1) \Rightarrow u = 4$$

$$x = 2 \Rightarrow u = 2^2 + 3(2) \Rightarrow u = 10$$

$$I = \int_1^2 \frac{2x + 3}{x^2 + 3x} dx$$

$$\therefore I = \int_4^{10} \frac{1}{u} du$$

$$\Rightarrow I = \left[ \ln |u| \right]_4^{10}$$

$$\Rightarrow I = \ln |10| - \ln |4|$$

$$\Rightarrow I = \ln(10/4)$$

$$\Rightarrow I = \ln(5/2); a = 5, b = 2$$

Example 10

Find the value of  $\int_1^3 \frac{1}{x^2 + 2x + 5} dx$  to three significant figures.

$$\text{Let } I = \int_1^3 \frac{1}{x^2 + 2x + 5} dx .$$

$$I = \int_1^3 \frac{1}{x^2 + 2x + 5} dx$$

$$\therefore I = \int_1^3 \frac{1}{(x + 1)^2 + 4} dx$$

$$\Rightarrow I = \int_1^3 \frac{1}{(x + 1)^2 + 2^2} dx$$

$$u = x + 1$$

$$\Rightarrow du = dx$$

$$x = 1 \Rightarrow u = 1 + 1 \Rightarrow u = 2$$

$$x = 3 \Rightarrow u = 3 + 1 \Rightarrow u = 4$$

$$I = \int_1^3 \frac{1}{(x + 1)^2 + 2^2} dx$$

$$\therefore I = \int_2^4 \frac{1}{u^2 + 2^2} du$$

$$\Rightarrow I = \left[ (1/2) \tan^{-1}(u/2) \right]_2^4$$

$$\Rightarrow I = (1/2) \tan^{-1}(4/2) - (1/2) \tan^{-1}(2/2)$$

$$\Rightarrow I = (1/2) \tan^{-1}(2) - \pi/8$$

$$\Rightarrow I = 0.1608\dots$$

$$\therefore I = 0.161 \text{ (3 s.f.)}$$

Example 11

Use the substitution  $u = 2 + \sqrt{x}$  to show that,

$$\int_0^4 \frac{\sqrt{x}}{2 + \sqrt{x}} dx = 8 \ln 2 - 4$$

$$\text{Let } I = \int_0^4 \frac{\sqrt{x}}{2 + \sqrt{x}} dx.$$

$$\begin{aligned} u &= 2 + \sqrt{x} \\ \Rightarrow du &= \frac{1}{2\sqrt{x}} dx \\ \Rightarrow du &= \frac{1}{2(u-2)} dx \\ \Rightarrow 2(u-2)du &= dx \\ x = 0 \Rightarrow u &= 2 + \sqrt{0} \Rightarrow u = 2 \\ x = 4 \Rightarrow u &= 2 + \sqrt{4} \Rightarrow u = 4 \end{aligned}$$

$$I = \int_0^4 \frac{\sqrt{x}}{2 + \sqrt{x}} dx$$

$$\therefore I = \int_2^4 \frac{(u-2)}{u} \cdot 2(u-2) du$$

$$\Rightarrow I = 2 \int_2^4 \frac{(u^2 - 4u + 4)}{u} du$$

$$\Rightarrow I = 2 \int_2^4 (u - 4 + 4u^{-1}) du$$

$$\Rightarrow I = 2 \left[ (1/2)u^2 - 4u + 4 \ln |u| \right]_2^4$$

$$\Rightarrow I = 2(8 - 16 + 4 \ln |4|) \\ - 2(2 - 8 + 4 \ln |2|)$$

$$\Rightarrow I = -16 + 8 \ln 4 + 12 - 8 \ln 2$$

$$\Rightarrow I = 8 \ln(4/2) - 4$$

$$\Rightarrow \boxed{I = 8 \ln 2 - 4}$$

## AH Maths - MiA (2<sup>nd</sup> Edn.)

- pg. 103 Ex. 7.2 Q 1a - e, g - t, v, w, 2, 3a, b, 4a, b (i).
- pg. 105-6 Ex. 7.3 Q 1c, d, 2c, d, 3, 4c, d, 5, 6c, d, 7 - 9, 11.
- pg. 107 Ex. 7.4 Q 1a - i, k - o, 2.
- pg. 109 Ex. 7.5 Q 1, 3, 5.
- pg. 111 Ex. 7.6 Q 1 - 5, 6b, 7.

**Ex. 7.2****1** Find these indefinite integrals.

**a**  $\int 3x^2(x^3 + 3)^4 dx$

**b**  $\int x(x^2 + 1)^5 dx$

**c**  $\int (2x + 1)(x^2 + x + 1)^6 dx$

**d**  $\int (x + 7)(x^2 + 14x)^{\frac{2}{3}} dx$

**e**  $\int \frac{2x}{x^2 + 1} dx$

**g**  $\int 3x \sqrt[3]{x^2 + 1} dx$

**h**  $\int \frac{1}{x} \ln x dx$

**i**  $\int \frac{1}{x \ln x} dx$

**j**  $\int \sin x e^{\cos x} dx$

**k**  $\int (x + 2)e^{x^2 + 4x} dx$

**l**  $\int \frac{\sin(\ln x)}{x} dx$

**m**  $\int \sin x \cos^4 x dx$

**n**  $\int \tan^3 x \sec^2 x dx$

**o**  $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

**p**  $\int (3x + 1)\sqrt{3x^2 + 2x + 1} dx$

**q**  $\int 3e^x(e^x + 1)^3 dx$

**r**  $\int \frac{4e^{2x}}{3e^{2x} - 2} dx$

**s**  $\int \cot x dx$  [ Remember:  $\cot x = \frac{\cos x}{\sin x}$  ]

**t**  $\int \frac{2x + 5}{x^2 + 5x - 1} dx$

**v**  $\int \frac{4x + 9}{(x + 4)(2x + 1)} dx$

**w**  $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

**2** **a** Find the derivative of  $\ln(\cos x)$ .**3** **a** Differentiate  $\cot x$ .

**b** Hence find  $\int \tan x \ln(\cos x) dx$ .

**b** Hence find  $\int \cosec^2 x \cot^3 x dx$ .

**4** **a** Show that  $\frac{1}{1 + e^x} = \frac{e^{-x}}{1 + e^{-x}}$ .

**b** Hence find **i**  $\int \frac{1}{1 + e^x} dx$

**Ex. 7.3**

- 1** Use the method shown in Example 6 to find the integrals.

c  $\int \sqrt{5-x^2} dx$       d  $\int \sqrt{1-2x^2} dx$

- 2** Use the substitution  $x = a \sin u$ , where  $a$  is an appropriate constant, and integrate these.

c  $\frac{3}{\sqrt{25-x^2}}$       d  $\frac{1}{\sqrt{1-2x^2}}$

- 3** a Use the substitution  $u = x^2$  to transform  $\int \frac{2x}{\sqrt{1-x^4}} dx$ .

b Now use the substitution  $u = \sin v$  to help you complete the integration.

- 4** Use the substitution  $x = a \tan u$ , where  $a$  is an appropriate constant, and integrate these.

c  $\frac{3}{x^2+16}$       d  $\frac{1}{1+4x^2}$

- 5** a Use the substitution  $u = x^4$  to transform  $\int \frac{x^3}{1+x^8} dx$ .

b Now use the substitution  $u = \tan v$  to help you complete the integration.

c Find  $\int \frac{\sin x}{1+\cos^2 x} dx$  by using the substitution  $u = \cos x$ .

- 6** Use the method shown in Example 8 to find the integrals.

c  $\int \frac{x}{2} \sqrt{16-x^2} dx$       d  $\int x \sqrt{1-3x^2} dx$

- 7** a Find  $\int \sin x \cos^2 x dx$  by letting  $u = \cos x$ .

b Find  $\int \sin^5 x \cos^2 x dx$  by letting  $u = \cos x$ .

c Find  $\int \sin^3 x \cos^5 x dx$  by letting  $u = \sin x$ .

d Find  $\int \sin x \cos x dx$  by letting  $u = \sin x$ .

e Find  $\int \cos^3 x dx$  by letting  $u = \sin x$ .

f Find  $\int \sin^5 x dx$  by letting  $u = \cos x$ .

- 8** a Given  $u = 1 - \sqrt{x}$ , show that  $dx = 2(u-1) du$ .

b Hence find  $\int \frac{2}{1-\sqrt{x}} dx$ .

c Similarly find  $\int \frac{2}{1+\sqrt{x}} dx$ .

- 9** a Use the substitution  $x = \tan u$  to transform  $\int \frac{1}{x^2 \sqrt{1+x^2}} dx$ .

b Now use the substitution  $v = \sin u$  to complete the integration.

- 11** Make use of the laws of logs and the substitution  $u = \ln x$  to help with these integrals.

a  $\int \frac{\ln(x^2)}{x} dx$       b  $\int \frac{\ln \sqrt{x}}{x} dx$       c  $\int \frac{\ln \sqrt{3x}}{2x} dx$       d  $\int \frac{x^2 + \ln(x^2)}{x} dx$

**Ex. 7.4**

- 1** Evaluate these definite integrals using the substitution suggested.

a  $\int_0^2 (x-2)(x^2-4x-5)^2 dx$  [let  $u = x^2 - 4x - 5$ ]

b  $\int_{-1}^1 3x^2(1-x^3)^3 dx$  [let  $u = 1-x^3$ ]

c  $\int_0^1 x\sqrt{1-x^2} dx$  [let  $u = 1-x^2$ ]

d  $\int_3^4 (2x+1)\sqrt{(x-3)(x+4)} dx$  [let  $u = (x-3)(x+4)$ ]

e  $\int_0^1 \frac{x+1}{x^2+2x+3} dx$  [let  $u = x^2 + 2x + 3$ ]

f  $\int_1^9 \frac{2\sqrt{x}+3}{\sqrt{x}} dx$  [let  $u = 2\sqrt{x} + 3$ ]

g  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^3 x} dx$  [let  $u = \sin x$ ]

h  $\int_{\ln 2}^{\ln 5} e^x \sqrt{e^x - 1} dx$  [let  $u^2 = e^x - 1$ ]

i  $\int_e^{e^2} \frac{dx}{x \ln x^3}$  [let  $u = \ln x$ ]

k  $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx$  [let  $u = \sin x$ ]

l  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$  [let  $u = \sin x$  then let  $u = \tan v$ ]

m  $\int_0^{\frac{\pi}{3}} \sin^7 x dx$  [let  $u = \cos x$ ]

n  $\int_0^{\frac{\pi}{4}} \cos^5 x dx$  [let  $u = \sin x$ ]

o  $\int_0^{\frac{\pi}{4}} \sec^2 x \sqrt{3 \tan x + 1} dx$  [let  $u = \tan x$ ]

- 2** Find these integrals, evaluating where limits are given.

Suitable substitutions have been suggested.

a  $\int_e^{e^3} \frac{(2+\ln x)^3}{x} dx$  [u = 2 + ln x] b  $\int_0^{\frac{\pi}{16}} \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$  [u<sup>2</sup> = x]

c  $\int \frac{x}{\sqrt{3+2x}} dx$  [u<sup>2</sup> = 3 + 2x] d  $\int \frac{3 dx}{x(5 \ln x + 4)^2}$  [u = 5 ln x + 4]

e  $\int_4^9 \frac{2}{3+\sqrt{x}} dx$  [u - 3 =  $\sqrt{x}$ ] f  $\int 2 \sin x (2-3 \cos x)^5 dx$  [u = 2 - 3 cos x]

g  $\int \frac{3}{\sqrt{9-x^2}} dx$  [x = 3 sin u] h  $\int_5^{10} \frac{x-5}{x-4} dx$  [u = x - 4]

i  $\int \frac{e^{\frac{1}{x}}}{x^2} dx$  [u = e<sup>1/x</sup>] j  $\int_0^{\sqrt{3}} \frac{x}{(4-x^2)^{\frac{3}{2}}} dx$  [u<sup>2</sup> = 4 - x<sup>2</sup>]

**Ex. 7.5**

**1** State the integrals of each of these.

$$\begin{array}{lll} \text{a} \int \sqrt[3]{3x-4} dx & \text{b} \int \sin(3-7x) dx & \text{c} \int \sec^2(2x-5) dx \\ \text{e} \int (8x+1)^{-1} dx & \text{f} \int (\sin(2x+1) + \cos(1-x)) dx & \text{g} \int (e^{-x} + (3-x)^{-1}) dx \end{array}$$

**3** State the integrals of each of these.

$$\begin{array}{lll} \text{a} \int (2x+3)(x^2+3x+4) dx & \text{b} \int \sin x \cos x dx & \text{c} \int \frac{\ln x}{x} dx \\ \text{d} \int e^x(e^x+1) dx & \text{e} \int \tan x \sec^2 x dx & \text{f} \int (\cos x + \sin x)(\cos x - \sin x) dx \\ \text{g} \int (e^x + e^{-x})(e^x - e^{-x}) dx & \text{h} \int (x^2 + \tan x)(2x + \sec^2 x) dx \end{array}$$

**5** Integrate

$$\begin{array}{lll} \text{a} \frac{3x^2 + 4x - 4}{x^3 + 2x^2 - 4x + 1} & \text{b} \frac{2 \cos 2x}{\sin 2x} & \text{c} \frac{3}{x+1} + \frac{2}{2x-1} \\ \text{e} \frac{\sec^2 x}{\tan x} & \text{f} \frac{x^{-1}}{\ln x} & \text{g} \frac{3^x \ln 3}{3 + 3^x} \\ \text{h} \frac{x}{x^2 - 1} + \frac{x^2}{x^3 - 1} + \frac{x^3}{x^4 - 1} \end{array}$$

**Ex. 7.6****1** Find

**a**  $\int \frac{dx}{\sqrt{16-x^2}}$

**b**  $\int \frac{dx}{\sqrt{3-x^2}}$

**c**  $\int \frac{dx}{\sqrt{32-2x^2}}$

**d**  $\int \frac{dx}{\sqrt{5-3x^2}}$

**2** Find

**a**  $\int \frac{dx}{49+x^2}$

**b**  $\int \frac{dx}{6+x^2}$

**c**  $\int \frac{dx}{3x^2+75}$

**d**  $\int \frac{dx}{5+2x^2}$

**3** Evaluate

**a**  $\int_0^3 \frac{dx}{\sqrt{36-x^2}}$

**b**  $\int_0^1 \frac{dx}{\sqrt{3-x^2}}$

**c**  $\int_{-1}^1 \frac{dx}{\sqrt{12-3x^2}}$

**d**  $\int_0^1 \frac{dx}{\sqrt{3-2x^2}}$

**4** Find correct to 3 dp where appropriate

**a**  $\int_{-3}^3 \frac{dx}{81+x^2}$

**b**  $\int_0^2 \frac{dx}{5+x^2}$

**c**  $\int_5^{10} \frac{dx}{4x^2+100}$

**d**  $\int_0^6 \frac{dx}{7+3x^2}$

**5** Find

**a**  $\int \frac{x-5}{25+x^2} dx$

**b**  $\int \frac{7-x}{16+x^2} dx$

**c**  $\int \frac{x-2}{x^2+2} dx$

**d**  $\int \frac{3x-1}{x^2+49} dx$

**6** Find

**b i**  $\int \frac{dx}{x^2+6x+25}$

**ii**  $\int \frac{dx}{x^2+4x+13}$

**iii**  $\int \frac{dx}{x^2+8x+18}$

**7** Evaluate to 3 sf

**a**  $\int_1^4 \frac{dx}{x^2+4x+13}$

**b**  $\int_0^1 \frac{dx}{x^2+8x+52}$

**c**  $\int_{-1}^1 \frac{dx}{x^2+4x+26}$

## Answers to AH Maths (MiA), pg. 103, Ex. 7.2

- |  |  |          |   |
|--|--|----------|---|
| <b>1 a</b>   | $\frac{1}{5}(x^3 + 3)^5 + c$                     | <b>b</b> | $\frac{1}{12}(x^2 + 1)^6 + c$               |
| <b>c</b>   | $\frac{1}{7}(x^2 + x + 1)^7 + c$                 | <b>d</b> | $\frac{3}{10}(x^2 + 14x)^{\frac{5}{3}} + c$ |
| <b>e</b>   | $\ln x^2 + 1  + c$                               |          |   |
| <b>g</b>   | $\frac{9}{8}(x^2 + 1)^{\frac{4}{3}} + c$         | <b>h</b> | $\frac{1}{2}(\ln x)^2 + c$                  |
| <b>i</b>   | $\ln \ln x   + c$                                |          |   |
| (hint: let $\ln x = u \Rightarrow e^u = x$ and $e^u du = dx$ ) |  |          |   |
| <b>j</b>   | $-e^{\cos x} + c$                                | <b>k</b> | $\frac{1}{2}e^{x^2+4x} + c$                 |
| <b>l</b>   | $-\cos(\ln x ) + c$                              | <b>m</b> | $-\frac{1}{5}\cos^5 x + c$                  |
| <b>n</b>   | $\frac{1}{4}\tan^4 x + c$                        | <b>o</b> | $2\sqrt{\tan x} + c$                        |
| <b>P</b>   | $\frac{1}{3}(3x^2 + 2x + 1)^{\frac{3}{2}} + c$   | <b>q</b> | $\frac{3}{4}(e^x + 1)^4 + c$                |
| <b>r</b>   | $\frac{2}{3}\ln 3e^{2x} - 2  + c$                | <b>s</b> | $\ln \sin x  + c$                           |
| <b>t</b>   | $\ln x^2 + 5x - 1  + c$                          |          |   |
| <b>v</b>   | $\ln (x + 4)(2x + 1)  + c$                       | <b>w</b> | $\ln e^x + e^{-x}  + c$                     |
| <b>2 a</b>   | $-\tan x$  | <b>b</b> | $-\frac{1}{2}(\ln \cos x )^2 + c$           |
| <b>3 a</b>   | $-\operatorname{cosec}^2 x$                      | <b>b</b> | $-\frac{1}{4}\cot^4 x + c$                  |
| <b>4 a</b>   | Multiply numerator and denominator by $e^{-x}$ . |          |   |
| <b>b i</b>   | $-\ln 1 + e^{-x}  + c$                           |          |   |

## Answers to AH Maths (MiA), pg. 105-6, Ex. 7.3

**1 c**  $\frac{1}{2}x\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

**d**  $\frac{1}{2}x\sqrt{1-2x^2} + \frac{\sqrt{2}}{4}\sin^{-1}(x\sqrt{2}) + c$

**2 c**  $3\sin^{-1}\left(\frac{x}{5}\right) + c$       **d**  $\frac{1}{\sqrt{2}}\sin^{-1}(x\sqrt{2}) + c$

**3 a**  $\int \frac{du}{\sqrt{1-u^2}}$       **b**  $\sin^{-1}(x^2) + c$

**4 c**  $\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$       **d**  $\frac{1}{2}\tan^{-1}(2x) + c$

**5 a, b**  $\frac{1}{4}\tan^{-1}(x^4) + c$       **c**  $-\tan^{-1}(\cos x) + c$

**6 c**  $-\frac{1}{6}(16-x^2)^{\frac{3}{2}} + c$       **d**  $-\frac{1}{9}(1-3x^2)^{\frac{3}{2}} + c$

**7 a**  $-\frac{1}{3}\cos^3 x + c$

**b**  $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$

**c**  $\frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + c$

**d**  $\frac{1}{2}\sin^2 x + c$

**e**  $\sin x - \frac{1}{3}\sin^3 x + c$

**f**  $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

**8 a**  $u = 1 - \sqrt{x} \Rightarrow \sqrt{x} = 1 - u \Rightarrow du = -\frac{1}{2\sqrt{x}}dx$   
 $\Rightarrow -2\sqrt{x}du = dx$ , hence result.

**b**  $4(1 - \sqrt{x}) - 4\ln|1 - \sqrt{x}| + c$

**c**  $4(1 + \sqrt{x}) - 4\ln|1 + \sqrt{x}| + c$

**9 a/b** 
$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\cos u du}{\sin^2 u} = \int \frac{d\nu}{\nu^2} = -\frac{1}{\nu} = -\frac{1}{\sin u}$$
  
 $= -\frac{1}{\sin(\tan^{-1}x)} + c$

**11 a**  $(\ln|x|)^2$

**b**  $\frac{1}{4}(\ln|x|)^2$

**c**  $\frac{\ln 3}{4}\ln|x| + \frac{1}{8}(\ln|x|)^2$

**d**  $\frac{x^2}{2} + (\ln|x|)^2$

## Answers to AH Maths (MiA), pg. 107, Ex. 7.4

1 a  $-100\frac{2}{3}$

b 4

c  $\frac{1}{3}$

d  $\frac{32}{3}\sqrt{2}$

e  $\ln \sqrt{2}$

f 28

g  $\frac{1}{2}$

h  $\frac{14}{3}$

i  $\frac{1}{3} \ln 2$

k  $\frac{2}{15}$

l  $\frac{\pi}{4}$

m  $\frac{289}{4480}$

n  $\frac{43}{60\sqrt{2}}$

o  $\frac{14}{9}$

2 a  $\frac{175}{4}$

b 2

c  $\frac{1}{6}(3 + 2x)^{\frac{3}{2}} - \frac{3}{2}(3 + 2x)^{\frac{1}{2}} + c$

d  $-\frac{3}{5(5 \ln x + 4)} + c$

e  $4 - 12 \ln \left(\frac{6}{5}\right)$

f  $\frac{1}{9}(2 - 3 \cos x)^6 + c$

g  $3 \sin^{-1} \left(\frac{x}{3}\right) + c$

h  $5 - \ln 6$

i  $-e^{\frac{1}{x}} + c$

j  $\frac{1}{2}$

**Answers to AH Maths (MiA), pg. 109, Ex. 7.5**

- |            |  |          |                                      |
|------------|--|----------|--------------------------------------|
| <b>1 a</b> | $\frac{3}{12}(3x - 4)^{\frac{4}{3}} + c$   | <b>b</b> | $\frac{1}{7} \cos(3 - 7x) + c$       |
| <b>c</b>   | $\frac{1}{2} \tan(2x - 5) + c$   | <b>d</b> | $-\frac{1}{6} e^{1-6x} + c$          |
| <b>e</b>   | $\frac{1}{8} \ln 8x + 1  + c$  |          |                                      |
| <b>f</b>   | $-\frac{1}{2} \cos(2x + 1) - \sin(1 - x) + c$  |          |                                      |
| <b>g</b>   | $-e^{-x} - \ln 3 - x  + c$   |          |                                      |
| <b>3 a</b> | $\frac{1}{2}(x^2 + 3x + 4)^2 + c$  | <b>b</b> | $\frac{1}{2}(\sin x)^2 + c$          |
| <b>c</b>   | $\frac{1}{2}(\ln x)^2 + c$   | <b>d</b> | $\frac{1}{2}(e^x + 1)^2 + c$         |
| <b>e</b>   | $\frac{1}{2}(\tan x)^2 + c$  | <b>f</b> | $\frac{1}{2}(\cos x + \sin x)^2 + c$ |
| <b>g</b>   | $\frac{1}{2}(e^x + e^{-x})^2 + c$  | <b>h</b> | $\frac{1}{2}(x^2 + \tan x)^2 + c$    |
| <b>5 a</b> | $\ln x^3 + 2x^2 - 4x + 1  + c$   | <b>b</b> | $\ln \sin 2x  + c$                   |
| <b>c</b>   | $3 \ln x + 1  + \ln 2x - 1  + c$   |          |                                      |
| <b>d</b>   | $\ln x + e^x  + c$   | <b>e</b> | $\ln \tan x  + c$                    |
| <b>f</b>   | $\ln \ln x  + c$   | <b>g</b> | $\ln 3 + 3^x  + c$                   |
| <b>h</b>   | $\frac{1}{2} \ln x^2 - 1  + \frac{1}{3} \ln x^3 - 1  + \frac{1}{4} \ln x^4 - 1  + c$ |          |                                      |

## Answers to AH Maths (MiA), pg. 111, Ex. 7.6

- 1 a**  $\sin^{-1}\left(\frac{x}{4}\right) + c$
- b**  $\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$
- c**  $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x}{4}\right) + c$
- d**  $\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{x}{\sqrt{\frac{5}{3}}}\right) + c$
- 2 a**  $\frac{1}{7} \tan^{-1}\left(\frac{x}{7}\right) + c$
- b**  $\frac{1}{\sqrt{6}} \tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + c$
- c**  $\frac{1}{15} \tan^{-1}\left(\frac{x}{5}\right) + c$
- d**  $\frac{1}{\sqrt{10}} \tan^{-1}\left(\frac{x\sqrt{2}}{\sqrt{5}}\right) + c$
- 3 a**  $\frac{\pi}{6}$
- b**  $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- c**  $\frac{\pi}{3\sqrt{3}}$
- d**  $\frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$
- 4 a** 0.072
- b** 0.326
- c** 0.016
- d** 0.288
- 5 a**  $\frac{1}{2} \ln |25 + x^2| - \tan^{-1}\left(\frac{x}{5}\right) + c$
- b**  $\frac{7}{4} \tan^{-1}\left(\frac{x}{4}\right) - \frac{1}{2} \ln |16 + x^2| + c$
- c**  $\frac{1}{2} \ln |x^2 + 2| - \sqrt{2} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$
- d**  $\frac{3}{2} \ln |x^2 + 49| - \frac{1}{7} \tan^{-1}\left(\frac{x}{7}\right) + c$
- 6 b** **i**  $\frac{1}{4} \tan^{-1}\left(\frac{x+3}{4}\right) + c$     **ii**  $\frac{1}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + c$   
**iii**  $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+4}{\sqrt{2}}\right) + c$
- 7 a** 0.107    **b** 0.0178    **c** 0.0765