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Unit 1 : Differential Equations - Lesson 2

Integrating Factor Differential Equations

LI

- Solve DEs that can be written in the form $y' + P(x)y = f(x)$.

SC

- Integrate.
- Product Rule in reverse.

Jan 10-12:11

A 1st-order linear differential equation is a differential equation that can be written in the form :

$$\frac{dy}{dx} + P(x)\,y = f(x) \quad ★$$

Solving a 1st-order linear differential equation requires the following steps :

- Find the integrating factor (IF) $= e^{\int P(x)\,dx}$.
- Multiply ★ by the IF.
- Integrate both sides wrt x and solve for y.

Aug 24-18:37

Example 1

Obtain the general solution of,

$$\frac{dy}{dx} - \frac{5}{x} y = 3x$$

$$P(x) = -\frac{5}{x}$$

$$\therefore \quad \int P(x)\, dx = -5 \int \frac{1}{x}\, dx$$

$$= -5 \ln |x| + C$$

$$IF = e^{\int P(x)\, dx}$$

$$\therefore \quad IF = e^{-5 \ln |x| + C}$$

$$\Rightarrow \quad IF = e^{C} e^{\ln |x|^{-5}}$$

$$\Rightarrow \quad \underline{IF = A x^{-5}} \quad (A = e^{C})$$

$$\frac{dy}{dx} - \frac{5}{x} y = 3x$$

$$\therefore \quad A x^{-5} \frac{dy}{dx} - \frac{5}{x} y \,.\, A x^{-5} = 3x \,.\, A x^{-5}$$

$$\Rightarrow \quad x^{-5} \frac{dy}{dx} - x^{-5} \frac{5}{x} y = 3 x^{-4}$$

$A = e^{C} > 0$;
so, $A \neq 0$

$$\Rightarrow \quad x^{-5} \frac{dy}{dx} \boxed{- 5 x^{-6}} y = 3 x^{-4}$$

'Opposite' of
Product Rule

$$\therefore \quad \frac{d}{dx} (x^{-5} y) = 3 x^{-4}$$

Integrating both sides wrt x gives,

$$x^{-5} y = -x^{-3} + C$$

$$\Rightarrow \quad \boxed{y = -x^{2} + C x^{5}}$$

Mar 1-14:16

Example 2

Obtain the general solution of,

$$y' \cos x + y \sin x = 1$$

Dividing by $\cos x$ gives,

$$y' + (\tan x)\, y = \sec x$$

$$P(x) = \tan x$$

$$\therefore \quad \int P(x)\, dx = \int \tan x \, dx$$

$$= -\ln|\cos x| + C$$

$$IF = e^{\int P(x)\, dx}$$

$$\therefore \quad IF = e^{-\ln|\cos x| + C}$$

$$\Rightarrow \quad IF = e^{C} e^{\ln|\cos x|^{-1}}$$

$$\Rightarrow \quad \underline{IF = A \sec x}$$

$$y' + (\tan x)\, y = \sec x$$

Multiplying by $A \sec x$ and cancelling A gives,

$$(\sec x)\, y' + (\sec x \tan x)\, y = \sec^2 x$$

$$\therefore \quad (\sec x \,.\, y)' = \sec^2 x$$

Integrating both sides wrt x gives,

$$\sec x \,.\, y = \tan x + C$$

$$\Rightarrow \quad \boxed{y = \sin x + C \cos x}$$

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Example 3

Obtain the general solution of,

$$y' + 7y = 4e^{-7x}$$

$$P(x) = 7$$

$$\therefore \quad \int P(x)\,dx = \int 7\,dx$$

$$= 7x + C$$

$$IF = e^{\int P(x)\,dx}$$

$$\therefore \quad IF = e^{7x + C}$$

$$\Rightarrow \quad IF = e^{C} e^{7x}$$

$$\Rightarrow \quad \underline{IF = A e^{7x}}$$

$$y' + 7y = 4e^{-7x}$$

Multiplying by $A e^{7x}$ and cancelling A gives,

$$\Rightarrow \quad e^{7x} y' + 7e^{7x} y = 4$$

$$\Rightarrow \quad (e^{7x} y)' = 4$$

Integrating both sides wrt x gives,

$$e^{7x} y = 4x + C$$

$$\Rightarrow \quad \boxed{y = 4x e^{-7x} + C e^{-7x}}$$

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AH Maths - MiA (2nd Edn.)

- pg. 136-7 Ex. 8.3 Q 1 a - e, g, k r, s, 2 a, b, d, e, f, 3 a, b, e.

Feb 10-15:47

Ex. 8.3

1 Solve each of these first-order linear differential equations.

a $\frac{dy}{dx} + y = e^x$

b $\frac{dy}{dx} + 3y = e^{-x}$

c $\frac{dy}{dx} + 4y = \sin x$

d $\frac{dy}{dx} + \frac{y}{x} = e^x$

e $\frac{dy}{dx} + \frac{3y}{x} = x^3$

g $\frac{dy}{dx} + \frac{y}{x-1} = x + 1$

k $\frac{dy}{dx} + y \sin x = 3 \sin x$

r $\frac{dy}{dx} - \frac{y}{x} = \frac{x}{x^2 + 1}$

s $\frac{dy}{dx} - \frac{y}{x} = \frac{x}{\sqrt{4 - x^2}}$

2 Find general solutions for these differential equations.

a $3\frac{dy}{dx} + y = 6e^x$

b $x\frac{dy}{dx} + y = \cos x$

d $\frac{1}{2}\frac{dy}{dx} + (x + 1)y = x + 1$

e $e^x\frac{dy}{dx} + y = 1$

f $\cos x\frac{dy}{dx} + 3y \sin x = 2$

3 Find the particular solution to each equation with the given initial conditions.

a $\frac{dy}{dx} + \frac{y}{2} = e^x$; when $x = 0, y = e$

b $\frac{dy}{dx} + \frac{y}{x} = \sin x$; when $x = \pi, y = 2$

e $\frac{dy}{dx} - y \ln x = x^x \sin x$; when $x = \pi, y = \pi^\pi$

Answers to AH Maths (MiA), pg. 136-7, Ex. 8.3

1 **a** $y = \frac{e^x}{2} + \frac{c}{e^x}$ **b** $y = \frac{1}{2e^x} + \frac{c}{e^{3x}}$

c $y = \frac{1}{17}(4\sin x - \cos x) + \frac{c}{e^{4x}}$

d $y = e^x - \frac{e^x}{x} + \frac{c}{x}$ **e** $y = \frac{x^4}{7} + \frac{c}{x^3}$

g $y = \frac{x^3 - 3x}{3(x-1)} + \frac{c}{x-1}$ **k** $y = 3 + ce^{\cos x}$

r $y = x\tan^{-1} x + xc$ **s** $y = x\sin^{-1}\left(\frac{x}{2}\right) + xc$

2 **a** $y = \frac{3}{2}e^x + ce^{-\frac{x}{3}}$ **b** $y = \frac{\sin x}{x} + \frac{c}{x}$

d $y = 1 + \frac{c}{e^{x^2+2x}}$ **e** $y = 1 + ce^{e^{-x}}$

f $y = \frac{2}{3}\sin x + \frac{4}{3}\sin x\cos^2 x + c\cos^3 x$

3 **a** $y = \frac{2}{3}e^x + \frac{e - \frac{2}{3}}{e^{\frac{x}{2}}}$ **b** $y = \frac{\sin x - x\cos x + \pi}{x}$

e $y = \frac{1}{2}x^x(\sin x - \cos x) + \frac{1}{2}e^{\pi - x}x^x$