Integration - Lesson 2

Integrals of More Complicated Expressions

LI

- Integrate expressions involving combinations of powers of x.

SC

- Sum and Difference Rules.
- Indices Rules.
- Expanding brackets.
- Splitting algebraic fractions.

Example 1

Integrate $y = \dfrac{1}{3x^4}$.

$$\int \frac{1}{3x^4}\,dx = \frac{1}{3}\int \frac{1}{x^4}\,dx$$

$$= \frac{1}{3}\int x^{-4}\,dx$$

$$= \frac{1}{3}\left(\frac{x^{-3}}{-3}\right) + C$$

$$\boxed{\begin{aligned} &= -\frac{x^{-3}}{9} + C \\ &\left(= -\frac{1}{9x^3} + C\right) \end{aligned}}$$

Example 2

Integrate $9\sqrt{x} - 10\sqrt{x^3}$.

$$\int 9\sqrt{x} - 10\sqrt{x^3}\,dx$$

$$= \int 9x^{1/2} - 10x^{3/2}\,dx$$

$$= \frac{9x^{3/2}}{3/2} - \frac{10x^{5/2}}{5/2} + C$$

$$= 6x^{3/2} - 4x^{5/2} + C$$

$$\left(= 6\sqrt{x^3} - 4\sqrt{x^5} + C\right)$$

Example 3

Integrate $x(x^2 + 7)$.

$$\int x(x^2 + 7)\,dx = \int x^3 + 7x\,dx$$

$$= \boxed{\frac{x^4}{4} + \frac{7x^2}{2} + C}$$

Example 4

Integrate $\dfrac{x - 1}{x\sqrt{x}}$.

$$\int \frac{x - 1}{x\sqrt{x}}\, dx$$

$$= \int \frac{x - 1}{x^{3/2}}\, dx$$

$$= \int \frac{x}{x^{3/2}} - \frac{1}{x^{3/2}}\, dx$$

$$= \int x^{-1/2} - x^{-3/2}\, dx$$

$$= \frac{x^{1/2}}{1/2} - \frac{x^{-1/2}}{-1/2} + C$$

$$\boxed{\begin{aligned} &= 2x^{1/2} + 2x^{-1/2} + C \\ &\left(= 2\sqrt{x} + \frac{2}{\sqrt{x}} + C \right) \end{aligned}}$$

Example 5

Prove that the integral of a non-zero constant function is a linear function.

Let $y = f(x)$ be a non-zero constant function, i. e. let,

$$y = k \qquad (k \neq 0)$$

$$\therefore \quad \int y \, dx = \int k \, dx$$

$$= kx + C$$

As the integral is of the form $kx + C$ $(k \neq 0)$, the integral is a linear function.

Example 6

Prove that the integral of a linear function is a quadratic function.

Let $y = f(x)$ be a linear function, i. e. let,

$$y = ax + b \qquad (a \neq 0)$$

$$\int y \, dx = \int ax + b \, dx$$

$$= \tfrac{1}{2} a x^2 + bx + C$$

As the integral is of the form $Ax^2 + bx + C$ $(A \neq 0)$, the integral is a quadratic function.

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Questions

1 For each of these expressions

i write the expression in integrable form

ii integrate with respect to x.

a $(2x-1)(x-3)$ **b** $x(x-4)(x+1)$ **c** $(x+2)(x^2+3x-4)$

d $5x^2(x-3)^2$ **e** $(x-2)(x+3)^2$

3 For each of these expressions

i write the expression in integrable form

ii integrate with respect to x.

a $\frac{6}{x^3}$ **b** $\frac{1}{5x^4}$ **c** $y=\frac{7}{3x^8}$ **d** $\frac{4}{x^2}-x^2+5$

4 For each of these expressions

i write the expression in integrable form

ii integrate with respect to x.

a $3\sqrt{x}$ **b** $\sqrt[3]{x^4}$ **c** $6\left(\sqrt[5]{x}\right)$ **d** $\frac{4}{\sqrt{x}}$

e $\frac{1}{x\sqrt{x}}$ **f** $\frac{3}{\sqrt[4]{x}}$ **g** $\frac{10}{\sqrt{x^5}}$ **h** $\frac{1}{2\left(\sqrt[4]{x^3}\right)}$

5 For each of the expressions:

i write the expression in integrable form

ii integrate with respect to x.

a $\frac{x^6-4}{x^2}$ **b** $\frac{9x-x^5}{x^4}$ **c** $\frac{x^4-x-3}{x^4}$

d $y=\frac{5-2x^4}{3x^2}$ **e** $\frac{(x-2)(x+3)}{x^4}$ **f** $\frac{(x-1)(3x+2)^2}{3x^5}$

6 Integrate with respect to x.

a $x\left(\sqrt{x}-4\right)$ **b** $\frac{2}{x}\left(x^2-\frac{1}{x}\right)$ **c** $\left(1-x^2\right)\left(2+\frac{1}{\sqrt{x}}\right)$

d $\left(\frac{5}{x}-\frac{x}{5}\right)^2$ **e** $\frac{1-x^3}{\sqrt{x}}$ **f** $\frac{(x+1)(2-x)}{x\sqrt{x}}$

7 Integrate with respect to the given variable.

a $\frac{2}{x^8}$ **b** $\frac{1}{5t^2}$ **c** $(p-1)(p+2)(p-4)$

d $6\left(\sqrt[3]{x^2}\right)$ **e** $\frac{2}{3\sqrt{w}}$ **f** $\frac{3-x^4}{x^3}$

g $2t^3(4-t)$ **h** $\frac{1}{u^2}-3\sqrt{u}+2$ **i** $\frac{3}{4\left(\sqrt[5]{x}\right)}$

Answers

1 **a** $2x^2 - 7x + 3$

$\frac{2x^3}{3} - \frac{7x^2}{2} + 3x + c$

b $x^3 - 3x^2 - 4x$

$\frac{x^4}{4} - x^3 - 2x^2 + c$

c $x^3 + 5x^2 + 2x - 8$

$\frac{x^4}{4} + \frac{5x^3}{3} + x^2 - 8x + c$

d $5x^4 - 30x^3 + 45x^2$

$x^5 - \frac{15x^4}{2} + 15x^3 + c$

e $x^3 + 4x^2 - 3x - 18$

$\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2} - 18x + c$

3 **a** $6x^{-3}$

$-\frac{3}{x^2} + c$

b $\frac{1}{5}x^{-4}$

$-\frac{1}{15}x^{-3} + c$

c $\frac{7}{3}x^{-8}$

$-\frac{1}{3x^7} + c$

d $4x^{-2} - x^2 + 5$

$-4x^{-1} - \frac{x^3}{3} + 5x + c$

4 **a** $3x^{\frac{1}{2}}$

$2x^{\frac{3}{2}} + c$

b $x^{\frac{4}{3}}$

$\frac{3x^{\frac{7}{3}}}{7} + c$

c $6x^{\frac{1}{5}}$

$5x^{\frac{6}{5}} + c$

d $4x^{\frac{-1}{2}}$

$8\sqrt{x} + c$

e $x^{\frac{-3}{2}}$

$-\frac{2}{\sqrt{x}} + c$

f $3x^{\frac{-1}{4}}$

$4x^{\frac{3}{4}} + c$

g $10x^{\frac{-5}{2}}$

$-\frac{20}{3x^{\frac{3}{2}}} + c$

h $\frac{1}{2}x^{\frac{-3}{4}}$

$2x^{\frac{1}{4}} + c$

5 **a** $x^4 - 4x^{-2}$

$\frac{4}{x} + \frac{x^5}{5} + c$

b $9x^{-3} - x$

$-\frac{9}{2x^2} - \frac{x^2}{2} + c$

c $1 - x^{-3} - 3x^{-4}$

$x + \frac{1}{2x^2} + \frac{1}{x^3} + c$

d $\frac{5}{3}x^{-2} - \frac{2}{3}x^2$

$\frac{1}{3}\left(-\frac{5}{x} - \frac{2x^3}{3}\right) + c$

e $x^{-2} + x^{-3} - 6x^{-4}$

$\frac{2}{x^3} - \frac{1}{2x^2} - \frac{1}{x} + c$

f $3x^{-2} + x^{-3} - \frac{8}{3}x^{-4} - \frac{4}{3}x^{-5}$

$\frac{1}{3x^4} + \frac{8}{9x^3} - \frac{1}{2x^2} - \frac{3}{x} + c$

6 **a** $-2x^2 + \frac{2x^{\frac{5}{2}}}{5} + c$

b $x^2 + \frac{2}{x} + c$

c $2\sqrt{x} + 2x - \frac{2}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}} + c$

d $-\frac{25}{x} - 2x + \frac{x^3}{75} + c$

e $2\sqrt{x} - \frac{2x^{\frac{7}{2}}}{7} + c$

f $-\frac{4}{\sqrt{x}} + 2\sqrt{x} - \frac{2x^{\frac{3}{2}}}{3} + c$

7 **a** $-\frac{2}{7x^7} + c$

b $-\frac{1}{5t} + c$

c $\frac{p^4}{4} - p^3 - 3p^2 + 8p + c$

d $\frac{18x^{\frac{5}{3}}}{5} + c$

e $\frac{4\sqrt{w}}{3} + c$

f $-\frac{3}{2x^2} - \frac{x^2}{2} + c$

g $-2\left(\frac{t^5}{5} - t^4\right) + c$

h $-2u^{\frac{3}{2}} + 2u - \frac{1}{u} + c$

i $\frac{15x^{\frac{4}{5}}}{16} + c$