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Vectors, Lines and Planes - Lesson 2

Equations of Planes

LI

- Know the different types of equations for a plane.
- Obtain equations of planes given different starting information.
- Convert between different forms of plane equations.

SC

- Vector product.
- Geometric intuition.

A plane is an infinite 2D space

An equation of a plane can be found in 3 ways :

- 1 point on the plane and a vector at right angles to the plane.
- 2 vectors in the plane and a point on the plane.
- 3 points on the plane.

A vector is parallel to a plane if it lies in the plane

A normal vector to a plane is one that is at right angles to the plane

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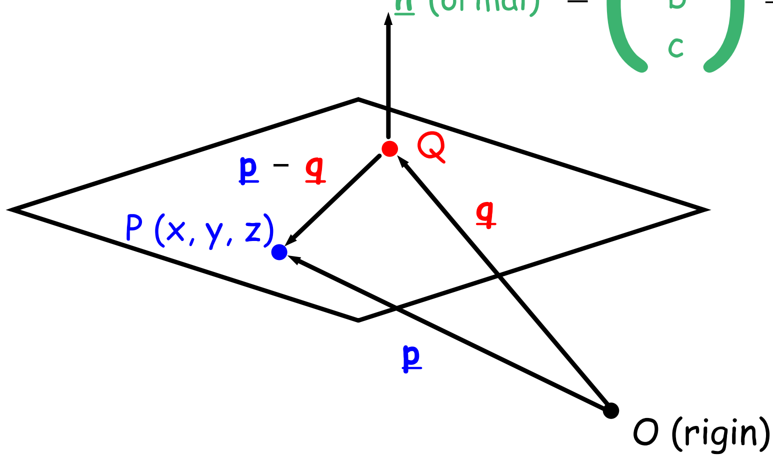
Cartesian Equation of a Plane

$$\underline{n}\text{ (ormal)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a, b, c)^T$$



The normal vector $\underline{n}$ is at right angles to $\underline{p} - \underline{q}$:

$$\underline{n} \bullet (\underline{p} - \underline{q}) = 0$$

$$\underline{n} \bullet \underline{p} = \underline{n} \bullet \underline{q}$$

Let $d = \underline{n} \bullet \underline{q}$. Then,

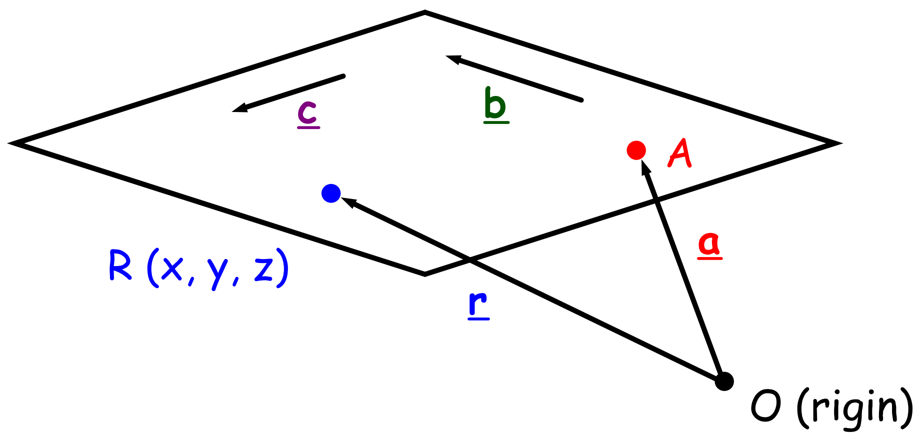
$$ax + by + cz = d$$

> The Cartesian equation of a plane with normal vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$
>
> and a point (x, y, z) on the plane is,
>
> $$ax + by + cz = d$$

If told 2 vectors $\underline{a}$ and $\underline{b}$ and a point on the plane, the vector product of $\underline{a}$ and $\underline{b}$ gives a normal.

If told 3 points A, B and C on the plane, work out two vectors (e.g. $\overrightarrow{AB}$ and $\overrightarrow{AC}$) and vector product these to give a normal.

Vector and Parametric Equations of a Plane



The vector equation of a plane (with parameters t and u) is,

$$\underline{r} = \underline{a} + t\,\underline{b} + u\,\underline{c}$$

'To get to $\underline{r}$, go to $\underline{a}$, then go a little bit along $\underline{b}$, then go a little bit along $\underline{c}$.'

If A has coordinates (a_1, a_2, a_3) and the vectors $\underline{b}$ and $\underline{c}$ have the obvious components, the vector equation can be written in parametric form :

$$\underline{r} = (a_1 + t\,b_1 + u\,c_1)\,\underline{i} + (a_2 + t\,b_2 + u\,c_2)\,\underline{j} + (a_3 + t\,b_3 + u\,c_3)\,\underline{k}$$

or

$$x = a_1 + t\,b_1 + u\,c_1$$

$$y = a_2 + t\,b_2 + u\,c_2$$

$$z = a_3 + t\,b_3 + u\,c_3$$

Example 1 (Cartesian equation given normal and 1 point)

Determine a Cartesian equation of the plane passing through the point $(2, 3, 1)$ and which has normal vector $(2, 0, -1)^T$.

$$ax + by + cz = d$$

$$\therefore \quad (2)(2) + (3)(0) + (1)(-1) = d$$

$$\Rightarrow \quad \underline{d = 3}$$

$$\therefore \quad 2 \, . \, x + 0 \, . \, y - 1 \, . \, z = 3$$

$$\Rightarrow \quad \boxed{2x - z = 3}$$

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Example 2 (Cartesian equation given 2 vectors and 1 point)

Find an equation of the plane passing through the point $(3, 1, 1)$ and containing the vectors $(2, 5, 1)^T$ and $(2, 1, -1)^T$.

A normal to the plane is found by taking the vector product of the given vectors lying in the plane :

$$\underline{\mathbf{n}} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$$

$$\therefore \quad (3)(-6) + (1)(4) + (1)(-8) = d$$

$$\Rightarrow \quad \underline{d = -22}$$

$$\therefore \quad -6x + 4y - 8z = -22$$

$$\Rightarrow \quad \boxed{3x - 2y + 4z = 11}$$

Example 3 (Cartesian equation given 3 points)

Determine an equation of the plane containing the points
A $(1, -6, 0)$, B $(-4, 2, -5)$ and C $(-2, 4, 1)$.

Two vectors lying in the plane are,

$$\overrightarrow{AB} = (-5, 8, -5)^T \text{ and } \overrightarrow{AC} = (-3, 10, 1)^T$$

A plane normal is (check !) $\underline{\mathbf{n}} = (58, 20, -26)^T$ and $d = -62$;
after simplification, an equation for the plane is,

$$\boxed{29x + 10y - 13z = -31}$$

Example 4 (Vector equation given normal and 1 point)

Determine a vector equation for the plane passing through the point (2, 3, 4) and which has normal $2\,\underline{\mathbf{i}} + 4\,\underline{\mathbf{k}}$.

Two vectors lying in the plane must be found; this is obtained by finding 2 solutions to the equation,

$$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By inspection, 2 solutions are $(-2, 0, 1)^T$ and $(0, 3, 0)^T$; thus, a vector equation is,

$$\boxed{\underline{\mathbf{r}} = (2, 3, 4) + t\,(-2, 0, 1)^T + u\,(0, 3, 0)^T}$$

Example 5

Find a Cartesian equation of the plane which has parametric form,

$$\underline{\mathbf{r}} = (3 - t + 2u)\,\underline{\mathbf{i}} + (1 + 7t - 4u)\,\underline{\mathbf{j}} + (1 - 5t + u)\,\underline{\mathbf{k}}$$

Rewriting the equation as,

$$\underline{\mathbf{r}} = (3, 1, 1) + t\,(-1, 7, -5)^T + u\,(2, -4, 1)^T$$

shows that the point $(3, 1, 1)$ lies in the plane and the vectors $(-1, 7, -5)^T$ and $(2, -4, 1)^T$ are parallel to the plane. A normal is thus found to be $(-13, -9, -10)^T$. A Cartesian equation is thus,

$$\boxed{13x + 9y + 10z = 58}$$

Example 6

Find parametric equations for the plane,

$$5x - 2y + 4z = 13$$

A normal is clearly $(5, -2, 4)^T$. By inspection, 2 vectors perpendicular to this normal (and hence lying in the plane) are $(2, 3, -1)^T$ and $(0, 2, 1)^T$. Hence,

$$\underline{r} = (1, -2, 1) + t\,(2, 3, -1)^T + u\,(0, 2, 1)^T$$

$$\Rightarrow \quad \underline{r} = (1 + 2t)\,\underline{i} + (-2 + 3t + 2u)\,\underline{j} + (1 - t + u)\,\underline{k}$$

$$\therefore \quad \boxed{\begin{aligned} x &= 1 + 2t \\ y &= -2 + 3t + 2u \\ z &= 1 - t + u \end{aligned}}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 291-2 Ex. 15.5
  Q 1 a, 2 b, 3 a, 4 b, 5 a, b, 10.
- pg. 296 Ex. 15.7
  Q 1 a, 2, 4 c, 6.

Ex. 15.5

1 Find the equation of the plane perpendicular to the given vector and containing the given point.

a $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$; A(0, 2, 6)

2 Find, in each case, the equation of the plane passing through P and perpendicular to PQ.

b P(3, −2, 1), Q (5, −7, 3)

3 a A plane is parallel to both the vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $4\mathbf{i} - 2\mathbf{k}$.

i By considering the cross product of these vectors, find a vector normal to the plane.

ii The plane contains the point (1, 1, 0). What is the equation of the plane?

4 Find the equation of the plane passing through the three points

b D(2, 1, 2), E(0, 3, −1) and F(3, 0, 4)

5 Find the equation of the plane passing through the first three points and say whether the fourth point is on the plane.

a A(3, 1, −4), B(2, −1, 2), C(−3, 2, 1) and D(1, 2, 3) b M(1, 0, 1), N(1, 1, 1), S(2, 1, −1) and T(1, 5, 1)

10 Find a unit vector which is normal to the plane which

a is parallel to $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

b passes through the points O(0, 0, 0), P(1, −1, 2) and Q(3, 2, −1).

Ex. 15.7

1 Find the equation of plane ABC in parametric form :

a $A(2, 1, 2)$, $B(0, 3, -1)$, $C(3, 0, 4)$ [Hint: first find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.]

2 A plane passes through the point $A(1, 1, 1)$ and is parallel to $2\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find the equation of this plane in parametric form.

4 Find the Cartesian equation of the plane whose parametric equations are

c $\mathbf{r} = (4 + t - 4u)\mathbf{i} + (-2 + t + 3u)\mathbf{j} + (1 + t - u)\mathbf{k}$

6 a Show that the point $P(-4, 15, 2)$ lies on the plane with parametric equation $\mathbf{r} = (2 - t + 3u)\mathbf{i} + (1 + 4t - 2u)\mathbf{j} + (2t + u - 3)\mathbf{k}$

b Show also that the point $Q(-5, 13, 5)$ does not lie on this plane.

Answers to AH Maths (MiA), pg. 291-2, Ex. 15.5

1 a $2x + 3y + z = 12$

2 b $2x - 5y + 2z = 18$

3 a $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ $\quad 2x - y + 4z = 1$

4 b $x + y = 3$

5 a $16x + 31y + 13z = 27$ no **b** $2x + z = 3$ yes

10 a $\frac{\pm 1}{\sqrt{398}}(-9\mathbf{i} - 11\mathbf{j} + 14\mathbf{k})$ **b** $\frac{\pm 1}{\sqrt{83}}(-3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})$

Answers to AH Maths (MiA), pg. 296, Ex. 15.7

1 a $\mathbf{r} = (2 - 2t + u)\mathbf{i} + (1 + 2t - u)\mathbf{j} + (2 - 3t + 2u)\mathbf{k}$

2 $\mathbf{r} = (1 + 2t + u)\mathbf{i} + (1 + t - u)\mathbf{j} + (1 + 2u)\mathbf{k}$

4 c $4x + 3y - 7z = 3$

6 a Verification, $t = 3$ $u = -1$ **b** Proof