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Vectors, Lines and Planes - Lesson 1

The Vector Product and the Scalar Triple Product

LI

- Calculate the Vector Product of 2 vectors.
- Calculate the Scalar Triple Product of 3 vectors.

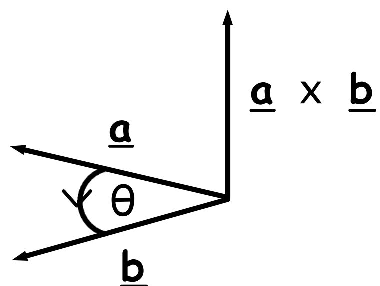
SC

- Vector Product Formulae.
- Determinants.

The vector product (aka cross product) of 2 vectors $\underline{a}$ and $\underline{b}$ is the vector,

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \; \underline{n} \qquad \text{(vector form)}$$

where $\underline{n}$ is a unit vector at right angles to both $\underline{a}$ and $\underline{b}$, and θ is the angle from $\underline{a}$ to $\underline{b}$. The 3 vectors $\underline{a}$, $\underline{b}$ and $\underline{a} \times \underline{b}$ form a right-handed system :



The component form of the vector product of $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ and $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$ is,

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Vector Product Properties

For the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$:

$\underline{i} \times \underline{j} = \underline{k}$ $\qquad$ $\underline{i} \times \underline{i} = \underline{0}$

$\underline{j} \times \underline{k} = \underline{i}$ $\qquad$ $\underline{j} \times \underline{j} = \underline{0}$

$\underline{k} \times \underline{i} = \underline{j}$ $\qquad$ $\underline{k} \times \underline{k} = \underline{0}$

For any vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$:

- $\underline{a} \times \underline{a} = \underline{0}$
- $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$
- $\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$
- $(\underline{a} + \underline{b}) \times \underline{c} = (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c})$

For any non-zero vectors $\underline{a}$ and $\underline{b}$:

$$\underline{a} \times \underline{b} = \underline{0}$$

$$\Updownarrow$$

$\underline{a}$, $\underline{b}$ are parallel ($\underline{a} = k\,\underline{b}$, $k \in \mathbb{R}$)

The scalar triple product of 3 vectors $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ (in that order !) is the scalar,

$$[\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}] = \underline{\mathbf{a}} \bullet (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

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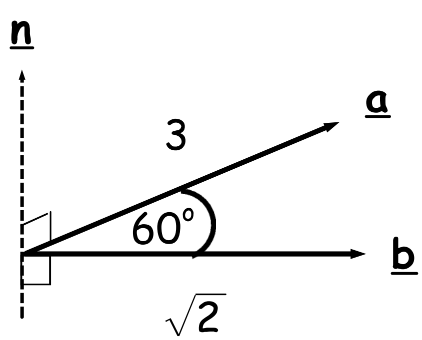
Example 1

Find $\underline{a} \times \underline{b}$ for the following :



($\underline{a}$ goes into the page)

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \ \underline{n}$$

$$\therefore \quad \underline{a} \times \underline{b} = 3\sqrt{2} \sin 300^\circ \ \underline{n}$$

$$\Rightarrow \quad \boxed{\underline{a} \times \underline{b} = -\frac{3\sqrt{3}\sqrt{2}}{2} \ \underline{n}}$$

Example 2

If $\underline{a} = 2\,\underline{i} - 5\,\underline{j} + 3\,\underline{k}$ and $\underline{b} = 4\,\underline{i} - 11\,\underline{j} - 7\,\underline{k}$, find $\underline{a} \times \underline{b}$.

$$\underline{a} \times \underline{b} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -11 \\ -7 \end{pmatrix}$$

$$\therefore \quad \underline{a} \times \underline{b} = \begin{pmatrix} (-5)(-7) - 3(-11) \\ (3)(4) - 2(-7) \\ (2)(-11) - (-5)(4) \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\underline{a} \times \underline{b} = \begin{pmatrix} 68 \\ 26 \\ -2 \end{pmatrix}}$$

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Example 3

Find a unit vector in the direction of $\underline{n}$ if $\underline{a}$, $\underline{b}$ and $\underline{n}$ form a right-handed system of vectors, where,

$$\underline{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

$$\underline{a} \times \underline{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \underline{a} \times \underline{b} = \begin{pmatrix} (1)(1) - (-1)(-1) \\ (-1)(1) - (1)(1) \\ (1)(-1) - (1)(1) \end{pmatrix}$$

$$\Rightarrow \quad \underline{a} \times \underline{b} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$$

$$\therefore \quad \underline{|\underline{a} \times \underline{b}| = \sqrt{8}}$$

A unit vector in the direction of $\underline{n}$ is,

$$\frac{1}{\sqrt{8}} \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = \boxed{-\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

Example 4

Show that no value of p makes the vectors $\underline{a} = 2p\,\underline{i} + \underline{j} - 3\,\underline{k}$ and $\underline{b} = -7\,\underline{i} + p\,\underline{j} - \underline{k}$ parallel.

$$\underline{a} = \begin{pmatrix} 2p \\ 1 \\ -3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -7 \\ p \\ -1 \end{pmatrix}.$$

$$\underline{a} \times \underline{b} = \begin{pmatrix} 2p \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -7 \\ p \\ -1 \end{pmatrix}$$

$$\therefore \quad \underline{a} \times \underline{b} = \begin{pmatrix} -1 + 3p \\ 21 + 2p \\ 2p^2 + 7 \end{pmatrix}$$

$$\boxed{2p^2 + 7 \neq 0 \ (p \in \mathbb{R}) \Rightarrow \underline{a} \times \underline{b} \neq \underline{0} \Rightarrow \underline{a}, \underline{b} \text{ not parallel}}$$

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Example 5

Find $[\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}]$ $(= \underline{\mathbf{b}} \bullet (\underline{\mathbf{a}} \times \underline{\mathbf{c}}))$ if $\underline{\mathbf{a}} = 3\,\underline{\mathbf{i}} + 2\,\underline{\mathbf{j}} - \underline{\mathbf{k}}$, $\underline{\mathbf{b}} = -2\,\underline{\mathbf{i}} + 5\,\underline{\mathbf{k}}$ and $\underline{\mathbf{c}} = -\,\underline{\mathbf{i}} + \underline{\mathbf{j}} + 4\,\underline{\mathbf{k}}$.

The vector product $\underline{\mathbf{a}} \times \underline{\mathbf{c}}$ can be evaluated first and then this 'dotted' with $\underline{\mathbf{b}}$. Alternatively, the determinant formula can be used.

$$[\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}] = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 5 \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix}$$

$$\therefore \quad [\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}] = -2\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$\Rightarrow \quad [\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}] = -2\,(8 + 1) - 0 + 5\,(3 + 2)$$

$$\Rightarrow \quad \boxed{[\underline{\mathbf{b}}, \underline{\mathbf{a}}, \underline{\mathbf{c}}] = 7}$$

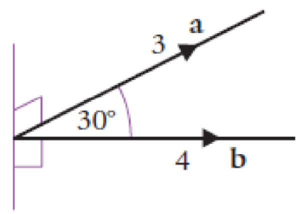
AH Maths - MiA (2nd Edn.)

- pg. 284-5 Ex. 15.2 Q 1 a, e, f.
- pg. 286-7 Ex. 15.3
  Q 1 a, b, 2 c, d, 8 c.
- pg. 288-9 Ex. 15.4 Q 1, 6.

Ex. 15.2

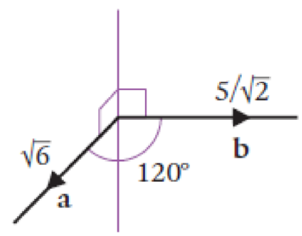
1 Copy each diagram and add $\mathbf{a} \times \mathbf{b}$ in each case, indicating the magnitude and direction clearly. The *normal* to the plane occupied by $\mathbf{a}$ and $\mathbf{b}$ is indicated in each case and a statement is included to clarify the ambiguity caused by optical illusion.

a



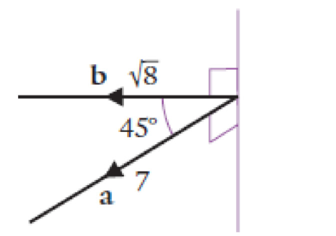
b is nearer

e



a comes out of the page

f



a comes out of the page

Ex. 15.3

1 Evaluate

a $\left|\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}\right|$ b $\left|\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix}\right|$

2 Express $\overrightarrow{PQ} \times \overrightarrow{PR}$ in component form for each set of coordinates.

c $P(0, 5, 3), Q(-1, 2, -1), R(6, 0, 0)$ d $P(-4\,0, 4), Q(0, 3, 2), R(-6, 5, -3)$

8 $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{n}$ form a right-handed system of vectors. Find the unit vector in the direction of $\mathbf{n}$ when $\mathbf{p}$ and $\mathbf{q}$ are given by

c $\mathbf{p} = \begin{pmatrix}2\\3\\0\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

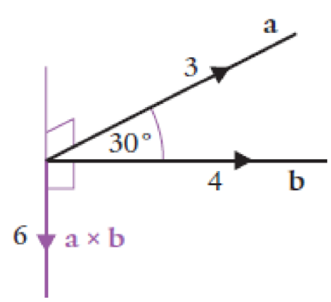
Ex. 15.4

1 By evaluating both expressions, verify that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

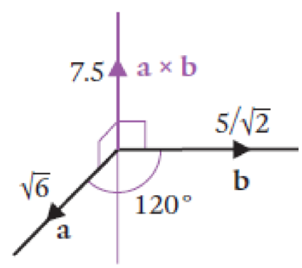
6 Given $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} + t\mathbf{k}$, for what value of t is $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = 33$?

Answers to AH Maths (MiA), pg. 284-5, Ex. 15.2

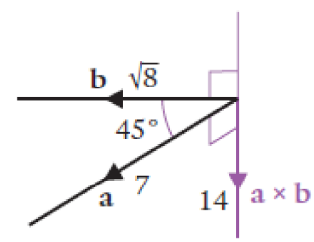
1 a



e



f



Answers to AH Maths (MiA), pg. 286-7, Ex. 15.3

1 a 1 b $3\sqrt{6}$

2 c $\begin{pmatrix} -11 \\ -27 \\ 23 \end{pmatrix}$ d $\begin{pmatrix} -11 \\ 32 \\ 26 \end{pmatrix}$

8 c $-\mathbf{k}$

Answers to AH Maths (MiA), pg. 288-9, Ex. 15.4

1 Verification, both equal 53.

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