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Unit 1 : Differential Equations - Lesson 1

Separable Differential Equations

LI

- Solve 1^{st}-Order Separable Ordinary Differential Equations.

SC

- Integration.

A differential equation (DE) is an equation involving derivatives of a function which is to be solved for the function

An n^{th}-order differential equation is a DE where the highest derivative is the n^{th} derivative

A separable differential equation is a DE that can be written in the form :

$$\frac{dy}{dx} = f(x)\,g(y)$$

The variables y and x are 'separated' from each other into the functions f and g

Differential equations are solved by integrating. The type of solution that is obtained depends on whether or not the integration constant is evaluated :

- General solution : constant not evaluated.
- Particular solution : constant evaluated.

To obtain a particular solution, initial conditions are required. This means, in the above notation, a value for x and a corresponding value for y.

How to Solve a Separable DE

Get into this form :
(or equivalent)

$$\frac{dy}{dx} = f(x)\,g(y)$$

Separate variables :

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

Integrate each side wrt relevant variable :

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

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Example 1

If $y > 0$, find the particular solution of the differential equation,

$$\cos^2 x \frac{dy}{dx} = y$$

with $y(\pi/4) = e^2$.

$$\cos^2 x \frac{dy}{dx} = y$$

$$\therefore \quad \int \frac{1}{y}\,dy = \int \frac{1}{\cos^2 x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \sec^2 x\,dx$$

$$\therefore \quad \ln|y| = \tan x + C$$

$$\Rightarrow \quad y = e^{\tan x + C}$$ $y > 0$ so $|y| = y$

$$\Rightarrow \quad y = e^{\tan x} . e^{C}$$

Writing $A = e^C$ this becomes,

$$\underline{y = A e^{\tan x}} \longleftarrow$$ General solution

Substituting in the initial conditions $x = \pi/4$, $y = e^2$ gives,

$$e^2 = A e^{\tan \pi/4}$$

$$\Rightarrow \quad e^2 = A e^1$$

$$\Rightarrow \quad \underline{A = e}$$

$$\therefore \quad y = e . e^{\tan x}$$

$$\Rightarrow \quad \boxed{y = e^{1 + \tan x}}$$

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Example 2

If $x, y > 0$, find the general solution of,

$$3y - x\frac{dy}{dx} = x^2\frac{dy}{dx}$$

$$3y - x\frac{dy}{dx} = x^2\frac{dy}{dx}$$

$$\Rightarrow \quad 3y = x^2\frac{dy}{dx} + x\frac{dy}{dx}$$

$$\Rightarrow \quad 3y = (x^2 + x)\frac{dy}{dx}$$

$$\Rightarrow \quad 3y = x(x + 1)\frac{dy}{dx}$$

$$\therefore \quad \int \frac{1}{y}\,dy = 3\int \frac{1}{x(x + 1)}\,dx$$

Using partial fractions (check !), this becomes,

$$\int \frac{1}{y}\,dy = 3\int \frac{1}{x}\,dx - 3\int \frac{1}{x + 1}\,dx$$

$$\therefore \quad \ln|y| = 3\ln|x| - 3\ln|x + 1| + C$$

$$\Rightarrow \quad \ln|y| = 3(\ln|x| - \ln|x + 1|) + C$$

$$\Rightarrow \quad \ln|y| = 3\ln\left|\frac{x}{x + 1}\right| + C$$

$$\Rightarrow \quad \ln|y| = \ln\left|\frac{x}{x + 1}\right|^3 + C$$

$$\Rightarrow \quad y = e^C\left(\frac{x}{x + 1}\right)^3 \qquad \text{As } x, y > 0$$

Making the usual replacement $A = e^C$ gives,

$$\boxed{y = A\left(\frac{x}{x + 1}\right)^3}$$

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Example 3

A colony consists of 300 people. A virus spreads, the number of people being infected after time t hours being denoted by $P(t)$ and governed by the differential equation,

$$\frac{dP}{dt} = P(300 - P)$$

Find $P(t)$ and explain what happens in the long run.

$$\frac{dP}{dt} = P(300 - P)$$

$$\therefore \quad \int dt = \int \frac{1}{P(300 - P)}\, dP$$

Partial fractions on the RHS (check !) gives,

$$\int dt = \frac{1}{300}\int \frac{1}{P}\, dP + \frac{1}{300}\int \frac{1}{300 - P}\, dP$$

$$\therefore \quad t + C = \frac{1}{300}\ln|P| - \frac{1}{300}\ln|300 - P|$$

Multiplying by 300, using the rules of logarithms, remembering that $P > 0$, relabelling e^{300C} as A, and using some heavy algebra, we obtain,

$$\boxed{P(t) = \frac{300\,A\,e^{300t}}{1 + A\,e^{300t}}}$$

Dividing top and bottom of the expression for $P(t)$ by e^{300t} gives,

$$P(t) = \frac{300\,A}{e^{-300t} + A}$$

In the 'long run' means as $t \longrightarrow \infty$. As $t \longrightarrow \infty$, $e^{-300t} \longrightarrow 0$. Hence, $P(t) \longrightarrow 300$. So,

> In the long run, the number of people in the colony infected by the virus approaches 300.

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AH Maths - MiA (2^{nd} Edn.)

- pg. 128-9 Ex. 8.1 Q 1 a, b, d - f, j, o - r, 2 c, i, l, m.
- pg. 131-4 Ex. 8.2 Q 1 a, b, 4, 7, 8.

Ex. 8.1

1 Find the general solution of each of these differential equations stating the solution explicitly, where possible.

a $\frac{dy}{dx} - 3x = 0$

b $x\frac{dy}{dx} - 1 = 0$

d $(1 + x^2)\frac{dy}{dx} = 1$

e $\frac{dy}{dx} - xe^x = 0$

f $\sec x\frac{dy}{dx} - \sin^2 x = 0$

j $\frac{dy}{dx} = (y + 1)(y - 2)$

o $5x\frac{dy}{dx} = \sqrt{1 - y^2}$

p $\frac{dy}{dx} = e^x \cos^2 y$

q $\frac{dy}{dx} = e^{x+y}$

r $\frac{dy}{dx} = xy(y - 1)$

2 Find the particular solution to each of these differential equations.

c $\frac{dy}{dx} = xe^x$ given that $y = e$ when $x = 1$

i $x^2\frac{dy}{dx} - y^2 = 0$ given that $y = 1$ when $x = 2$

l $\frac{x}{y}\frac{dy}{dx} = \ln y$ given that $y = e$ when $x = 2$
[Hint: use the substitution $u = \ln y$.]

m $e^x\frac{dy}{dx} = x\sqrt{y}$ given that $y = 4$ when $x = 0$

Ex. 8.2

1 Write a mathematical model in the form of a differential equation for each of these statements.

a The rate of change of displacement (s) with time (t) is directly proportional to the displacement.

b The rate of change of displacement (s) with time (t) is inversely proportional to the time.

4 A new TV series has a prospective target audience in mind. As the weeks go by after the launch of the series the percentage of the target audience watching the programme, V%, is expected to grow according to the model $\frac{dV}{dt} = ke^{-0.3t}$ where t is measured in weeks and k is a constant.

a Express V in terms of t and the constants k and c where c is the constant of integration.

b Initially none of the audience are watching (when $t = 0, V = 0$).

Express c in terms of k.

c After five weeks 77.7% of the target audience are watching.

Express V in terms of t.

d After 10 weeks 90% of the target audience are watching.

Is the model holding up? Comment.

7 A learner driver has to learn 400 facts for her test. The rate at which the number of facts she can recall grows is proportional to the number of facts still to memorise.

She applies herself regularly to the task on a daily basis. Her learning can be modelled using a second type of growth model, $\frac{dF}{dt} = k(400 - F)$ where F is the number of facts memorised and t is the time measured in days since she started the task. Initially she knew no facts. After five days she could memorise 250 facts.

a Express F explicitly in terms of t.

b When she can remember 80% of the facts she can claim to have mastery of the topic.

For how many days will she have to study to claim mastery?

8 In a third model, the *restricted* growth model, the quantity again has a limit to which it can grow. The rate of increase of the quantity at a particular time is jointly proportional to the quantity present at that time and the quantity needed to reach the limit.

In a small island there are 2000 inhabitants. An islander returns from his holidays and brings a flu virus with him. It spreads among the population according to the model $\frac{dP}{dt} = kP(2000 - P)$ where P is the number with the virus after t days and k is the constant of proportion.

a With the aid of partial fractions show that $\frac{1}{2000}\ln\left(\frac{P}{2000 - P}\right) = kt + c$ where k and c are constants.

b Assuming that $P = 1$ when $t = 0$, find the value of c to 1 significant figure.

c If 20 people have contracted the virus after five days

i express t in terms of P

ii express P in terms of t.

d Help will have to be flown in if more than 50% of the inhabitants contract the virus.

Estimate the number of days before this happens.

Answers to AH Maths (MiA), pg. 128-9, Ex. 8.1

1

a $y = \frac{3x^2}{2} + c$

b $y = \ln|x| + c$

d $y = \tan^{-1} x + c$

e $y = e^x x - e^x + c$

f $\frac{1}{3}\sin^3 x + c$

j $y = \frac{e^{3x+c} + 2}{1 - e^{3x+c}}$

o $y = \sin\left(\frac{1}{5}\ln x + c\right)$

p $y = \tan^{-1}(e^x + c)$

q $y = -\ln|c - e^x|$

r $y = \frac{1}{1 - e^{\frac{x^2}{2}+c}}$

2

c $y = e^x x - e^x + e$

i $y = \frac{2x}{2 + x}$

l $y = e^{\frac{x}{2}}$

m $y = \frac{1}{4}\left(-\frac{x}{e^x} - \frac{1}{e^x} + 5\right)^2$

Answers to AH Maths (MiA), pg. 131-4, Ex. 8.2

1 **a** $\frac{ds}{dt} = ks$ **b** $\frac{ds}{dt} = \frac{k}{t}$

4 **a** $V = -\frac{k}{0{\cdot}3} e^{-0{\cdot}3t} + c$ **b** $c = \frac{k}{0{\cdot}3}$

c $V = 100(1 - e^{-0{\cdot}3t})$ **d** No. 95% predicted.

7 **a** $F = 400\left(1 - \left(\frac{3}{8}\right)^{\frac{t}{5}}\right)$ **b** $t = 8{\cdot}2 \Rightarrow$ on 9th day

8 Accurate figures used: rounded for answer. p

a Use $\int \frac{1}{2000}\left[\frac{1}{P} + \frac{1}{2000 - P}\right] dP = \frac{1}{2000} \ln\left(\frac{P}{2000 - P}\right)$

b $c = -0{\cdot}004$

c **i** $t = \frac{1}{0{\cdot}6}\left[\ln\left(\frac{P}{2000 - P}\right)\right] + 13$ **ii** $P = \frac{2000e^{0{\cdot}6t - 7{\cdot}6}}{1 + e^{0{\cdot}6t - 7{\cdot}6}}$

d 13 days