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Unit 1 : Partial Fractions - Lesson 1

Rational Functions and Partial Fractions

LI

- Know the types of Rational Functions (RFs).
- Find Partial Fractions for specific types of RFs.

SC

- Algebra.

A polynomial divided by another polynomial is called a **rational function**,

$$\frac{p(x)}{q(x)}$$

If $\deg p < \deg q$, the above is called a **proper rational function**, whereas if $\deg p \geq \deg q$, the above is called an **improper rational function**.

A rational function can be written in terms of a proper rational function.

A polynomial p is **reducible** if it can be factorised into polynomials, none of which are equal to the polynomials 1 and p. e.g. $x^2 - 4$

A polynomial is **irreducible** if it is not reducible. e.g. $x^2 + 4$

<u>Theorem (**Partial Fraction Decomposition Theorem**):</u>

Any rational function $\frac{p}{q}$ can be written as a polynomial plus a sum of proper rational functions each of which is of the form,

$$\boxed{\frac{g(x)}{r(x)^n}} \qquad (n \in \mathbb{N})$$

where r is an irreducible factor of q and $\deg g < \deg r$; such proper rational functions are called **partial fractions** of $\frac{p}{q}$.

Types of Partial Fractions Arising from a Cubic Denominator

In the following table, $a \neq 0$, each factor is non-zero and $S, T \in \mathbb{R}$.

Factor	Partial Fraction
$ax + b$ (**non-repeated linear**)	$\frac{S}{ax+b}$
$(ax+b)^2$ (**repeated linear**)	$\frac{S}{ax+b} + \frac{T}{(ax+b)^2}$
$ax^2 + bx + c$ (**irreducible quadratic**)	$\frac{Sx+T}{ax^2+bx+c}$

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Example 1

Find partial fractions for $\dfrac{3x + 5}{(x + 1)(x + 2)(x - 3)}$.

$$\frac{3x + 5}{(x + 1)(x + 2)(x - 3)} = \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x - 3}$$

$$\boxed{3x + 5 = A(x + 2)(x - 3) + B(x + 1)(x - 3) + C(x + 1)(x + 2)}$$

$\underline{x = -1:}$

$$3(-1) + 5 = A(-1 + 2)(-1 - 3)$$

$$\Rightarrow \quad 2 = A(1)(-4)$$

$$\Rightarrow \quad \underline{A = -1/2}$$

$\underline{x = -2:}$

$$3(-2) + 5 = B(-2 + 1)(-2 - 3)$$

$$\Rightarrow \quad -1 = B(-1)(-5)$$

$$\Rightarrow \quad \underline{B = -1/5}$$

$\underline{x = 3:}$

$$3(3) + 5 = C(3 + 1)(3 + 2)$$

$$\Rightarrow \quad 14 = C(4)(5)$$

$$\Rightarrow \quad \underline{C = 7/10}$$

$$\therefore \boxed{\begin{aligned} \frac{3x + 5}{(x + 1)(x + 2)(x - 3)} &= \frac{(-1/2)}{x + 1} + \frac{(-1/5)}{x + 2} + \frac{7/10}{x - 3} \\ &= \frac{-1}{2(x + 1)} - \frac{1}{5(x + 2)} + \frac{7}{10(x - 3)} \end{aligned}}$$

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Example 2

Express $\dfrac{x^2 + 6x - 4}{(x + 2)^2(x - 4)}$ in partial fractions.

$$\frac{x^2 + 6x - 4}{(x + 2)^2 (x - 4)} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{x - 4}$$

$$\boxed{x^2 + 6x - 4 = A(x + 2)(x - 4) + B(x - 4) + C(x + 2)^2}$$

$\underline{x = 4:}$

$$4^2 + 6(4) - 4 = C(4 + 2)^2$$

$$\Rightarrow \quad 36 = C(36)$$

$$\Rightarrow \quad \underline{C = 1}$$

$\underline{x = -2:}$

$$(-2)^2 + 6(-2) - 4 = B(-2 - 4)$$

$$\Rightarrow \quad -12 = B(-6)$$

$$\Rightarrow \quad \underline{B = 2}$$

$\underline{\text{Coefficients of } x^2:}$

$$1 = A + C$$

$$\Rightarrow \quad \underline{A = 0}$$

$$\therefore \quad \boxed{\frac{x^2 + 6x - 4}{(x + 2)^2 (x - 4)} = \frac{2}{(x + 2)^2} + \frac{1}{x - 4}}$$

Example 3

Find partial fractions for $\dfrac{x^2 - 4}{(3x + 2)(x^2 + 1)}$.

$$\frac{x^2 - 4}{(3x + 2)(x^2 + 1)} = \frac{A}{3x + 2} + \frac{Bx + C}{x^2 + 1}$$

$$\boxed{x^2 - 4 = A(x^2 + 1) + (Bx + C)(3x + 2)}$$

$x = -2/3$:

$$(-2/3)^2 - 4 = A((-2/3)^2 + 1)$$
$$\Rightarrow \quad 4/9 - 4 = A(4/9 + 1)$$
$$\Rightarrow \quad -32/9 = A(13/9)$$
$$\Rightarrow \quad -32/9 = A(13/9)$$
$$\Rightarrow \quad A = -32/13$$

$$\boxed{\begin{aligned} x^2 - 4 &= A(x^2 + 1) + (Bx + C)(3x + 2) \\ x^2 - 4 &= Ax^2 + A + 3Bx^2 + 3Cx + 2Bx + 2C \\ x^2 - 4 &= (A + 3B)x^2 + (2B + 3C)x + (A + 2C) \end{aligned}}$$

Coefficients of x^2 :

$$A + 3B = 1$$
$$\Rightarrow \quad 3B = 1 + 32/13$$
$$\Rightarrow \quad 3B = 45/13$$
$$\Rightarrow \quad B = 15/13$$

Constant Terms :

$$A + 2C = -4$$
$$\Rightarrow \quad 2C = -4 + 32/13$$
$$\Rightarrow \quad 2C = -20/13$$
$$\Rightarrow \quad C = -10/13$$

$$\therefore \quad \boxed{\begin{aligned} \frac{x^2 - 4}{(3x + 2)(x^2 + 1)} &= \frac{(-32/13)}{3x + 2} + \frac{(15/13)x + (-10/13)}{x^2 + 1} \\ &= \frac{-32}{13(3x + 2)} + \frac{15x - 10}{13(x^2 + 1)} \end{aligned}}$$

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AH Maths - MiA (2nd Edn.)

- pg. 25-6 Ex. 2.4
  Q 16 - 22, 24, 25, 27 - 30, 32, 34 - 36, 39.

Resolve each proper rational function into its partial fractions.

16 $\dfrac{x}{(1-x)(2+x)}$

17 $\dfrac{2x-1}{(2x+1)(x-3)}$

18 $\dfrac{3x}{(x-2)(x+1)}$

19 $\dfrac{2}{(x-1)^2(x+1)}$

20 $\dfrac{3x^2-4}{x(x^2+1)}$

21 $\dfrac{3}{x(x-2)^2}$

22 $\dfrac{1}{x(x^2+4)}$

23 $\dfrac{4x-3}{x^3(x+1)}$

24 $\dfrac{5x-3}{(x+2)(x-3)^2}$

25 $\dfrac{3x^2+2x}{(x+2)(x^2+3)}$

26 $\dfrac{3}{1-x^3}$

27 $\dfrac{x^2+1}{x(x^2-1)}$

28 $\dfrac{2x-1}{(x-2)(x+1)(x+3)}$

29 $\dfrac{4x-1}{x^2(x^2-4)}$

30 $\dfrac{1}{x^2-2}$

31 $\dfrac{x^2}{(x-3)^2}$

32 $\dfrac{(x+13)^2}{(x-3)^2(x+5)}$

33 $\dfrac{1-2x}{x^3+1}$

34 $\dfrac{x}{x^4-16}$

35 $\dfrac{2x-1}{(x-3)^2(x+5)}$

36 $\dfrac{3x}{(x+1)(3-x^2)}$

37 $\dfrac{2x^2-5x}{(x^2-1)(x^2-4)}$

38 $\dfrac{1}{x^3(1-2x)}$

39 $\dfrac{2x-7}{(x^2+4)(x-1)^2}$

40 $\dfrac{1}{x(x^2-1)^2}$

41 $\dfrac{1}{x(x^2+4)^2}$

Answers to AH Maths (MiA), pg. 25-6, Ex. 2.4

16 $\frac{1}{3(1-x)} - \frac{2}{3(2+x)}$

17 $\frac{4}{7(2x+1)} + \frac{5}{7(x-3)}$

18 $\frac{2}{x-2} + \frac{1}{x+1}$

19 $\frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{1}{(x-1)^2}$

20 $\frac{7x}{x^2+1} - \frac{4}{x}$

21 $\frac{3}{4x} - \frac{3}{4(x-2)} + \frac{3}{2(x-2)^2}$

22 $\frac{1}{4x} - \frac{x}{4(x^2+4)}$

24 $\frac{12}{5(x-3)^2} + \frac{13}{25(x-3)} - \frac{13}{25(x+2)}$

25 $\frac{8}{7(x+2)} + \frac{13x-12}{7(x^2+3)}$

27 $\frac{1}{x-1} - \frac{1}{x} + \frac{1}{x+1}$

28 $\frac{1}{5(x-2)} + \frac{1}{2(x+1)} - \frac{7}{10(x+3)}$

29 $\frac{7}{16(x-2)} + \frac{9}{16(x+2)} - \frac{1}{x} + \frac{1}{4x^2}$

30 $\frac{\sqrt{2}}{4(x-\sqrt{2})} - \frac{\sqrt{2}}{4(x+\sqrt{2})}$

32 $\frac{1}{x+5} + \frac{32}{(x-3)^2}$

34 $\frac{1}{16(x-2)} + \frac{1}{16(x+2)} - \frac{x}{8(x^2+4)}$

35 $\frac{11}{64(x-3)} + \frac{5}{8(x-3)^2} - \frac{11}{64(x+5)}$

36 $\frac{3(\sqrt{3}-1)}{4(\sqrt{3}-x)} + \frac{3(\sqrt{3}+1)}{4(\sqrt{3}+x)} - \frac{3}{2(x+1)}$

39 $\frac{4}{5(x-1)} - \frac{1}{(x-1)^2} + \frac{1-4x}{5(x^2+4)}$