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Geometry of Complex Numbers - Lesson 1

Polar Form,Multiplication and Division

LI

- Change a complex number between Cartesian and Polar Form.
- Multiply and divide complex numbers in polar form.

SC

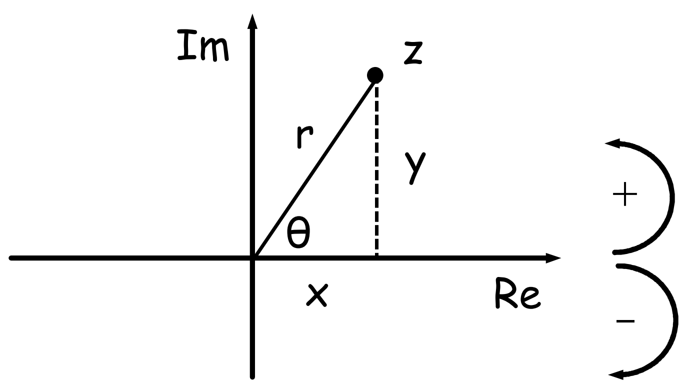
- Modulus and Argument.

The set of all complex numbers $\mathbb{C}$ can be geometrically described as the complex plane

An Argand diagram is a plot of one or more complex numbers in the complex plane

A complex number written in Cartesian form as $z = x + yi$ can be written in Polar Form :

$$z = r(\cos\theta + i\sin\theta)$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

Modulus of z : $r = |z| = \sqrt{x^2 + y^2}$

An Argument of z : $\operatorname{Arg} z = \theta$

z has infinitely many arguments (keep adding or subtracting multiples of 2π); the principal argument of z ($\arg z$) is the value of $\operatorname{Arg} z$ that lies in the range $-\pi < \theta \le \pi$.

The Polar form makes multiplication and division of complex numbers very easy

Given complex numbers z and w in polar form,

$$z = r(\cos\theta + i\sin\theta) \text{ and } w = R(\cos\varphi + i\sin\varphi)$$

the product and quotient can be formed and simplified (using addition formulae) to give,

$$zw = rR(\cos(\theta + \varphi) + i\sin(\theta + \varphi))$$

$$\frac{z}{w} = \frac{r}{R}(\cos(\theta - \varphi) + i\sin(\theta - \varphi))$$

To multiply two complex numbers in polar form, multiply the moduli and add the arguments :

$$|zw| = |z|\,|w|; \quad \operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$$

To divide two complex numbers in polar form, divide the moduli and subtract the arguments :

$$|z \div w| = |z| \div |w|; \quad \operatorname{Arg}(z \div w) = \operatorname{Arg} z - \operatorname{Arg} w$$

The resultant argument may need to be brought into the correct range ($-\pi < \theta \leq \pi$) by adding or subtracting multiples of 2π.

Useful shorthand : $\cos\theta + i\sin\theta = \operatorname{cis}\theta$

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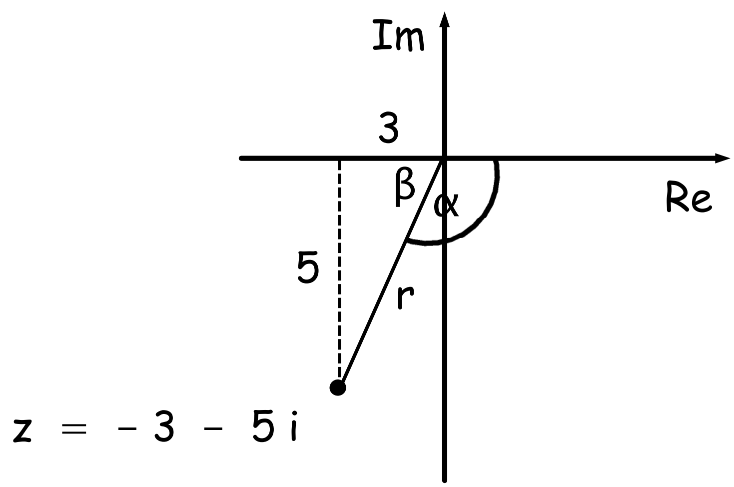
Example 1

Find the modulus and argument (in radians, to 3 s.f.) of $z = -3 - 5i$, and hence write z in polar form.



Always plot the complex number first

$r = \sqrt{x^2 + y^2}$

$\therefore \quad r = \sqrt{(-3)^2 + (-5)^2}$

$\Rightarrow \quad \boxed{r = \sqrt{34}}$

$\beta = \tan^{-1}(5/3)$

$\Rightarrow \quad \underline{\beta = 1.03\ldots}$

$\therefore \quad \alpha = \pi - 1.03\ldots$

$\Rightarrow \quad \underline{\alpha = 2.11\ldots}$

$\therefore \quad \boxed{\arg z = -2.11}$

$$\boxed{-3 - 5i = \sqrt{34}\,(\cos(-2.11) + i\sin(-2.11))}$$

Example 2

Express $z = 2 \operatorname{cis}(2\pi/3)$ in Cartesian form.

$$z = 2 \operatorname{cis}(2\pi/3)$$

$$\therefore \quad z = 2(\cos(2\pi/3) + i\sin(2\pi/3))$$

$$\Rightarrow \quad z = 2((-1/2) + i(\sqrt{3}/2))$$

$$\Rightarrow \quad \boxed{z = -1 + i\sqrt{3}}$$

Example 3

Simplify $7 \text{ cis } (\pi/6) \times 3 \text{ cis } (\pi/4)$ in polar form.

Let $z = 7 \text{ cis } (\pi/6) \times 3 \text{ cis } (\pi/4)$.

$$z = 7 \text{ cis } (\pi/6) \times 3 \text{ cis } (\pi/4)$$

$$\Rightarrow \quad z = 21 \text{ cis } (\pi/6 + \pi/4)$$

$$\Rightarrow \quad \boxed{z = 21 \text{ cis } (5\pi/12)}$$

Example 4

Simplify $2 \text{ cis } (\pi/3) \div 2 \text{ cis } (\pi/2)$, writing the answer in polar form and then Cartesian form.

Let $w = 2 \text{ cis } (\pi/3) \div 2 \text{ cis } (\pi/2)$.

$$w = 2 \text{ cis } (\pi/3) \div 2 \text{ cis } (\pi/2)$$

$$\Rightarrow \quad w = \text{cis } (\pi/3 - \pi/2)$$

$$\Rightarrow \quad \boxed{w = \text{cis } (-\pi/6)}$$

$$w = \cos(-\pi/6) + i \sin(-\pi/6)$$

$$\Rightarrow \quad \boxed{w = \frac{\sqrt{3}}{2} - \frac{1}{2} i}$$

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Example 5

Express z^4 in polar form (fully simplified) if $z = 2\text{ cis }(\pi/3)$.

$$z^4 = 2\text{ cis }(\pi/3) \times 2\text{ cis }(\pi/3) \times 2\text{ cis }(\pi/3) \times 2\text{ cis }(\pi/3)$$

$$\Rightarrow \quad z^4 = 16\text{ cis }(\pi/3 + \pi/3 + \pi/3 + \pi/3)$$

$$\Rightarrow \quad z^4 = 16\text{ cis }(4\pi/3)$$

The angle $4\pi/3$ is not in the required range $(-\pi < \theta \leq \pi)$ - it's bigger than π; we therefore keep on subtracting 2π to get it into the required range.

$$4\pi/3 - 2\pi = 4\pi/3 - 6\pi/3 = -2\pi/3$$

$$\boxed{z^4 = 16\text{ cis }(-2\pi/3)}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 211 Ex. 12.3 Q 1-4, 6, 7.
- pg. 215 - 6 Ex. 12.5
  Q 1, 2 a, c, 3.

Ex. 12.3

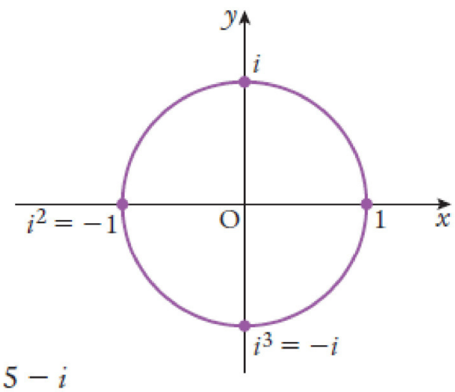
1 $z = 1 \Rightarrow |z| = 1$ and $\arg z = 0$

Use the diagram to help you make similar statements about

a $z = i$

b $z = -1$

c $z = -i$



2 a Draw Argand diagrams to illustrate

i $3 + 4i$ and $3 - 4i$ ii $2 + 3i$ and $2 - 3i$ iii $5 + i$ and $5 - i$

b Comment on the Argand diagrams of z and $\bar{z}$.

3 For each of these complex numbers

i plot the number on an Argand diagram

ii find the modulus and argument to 3 significant figures where appropriate.

a $1 + i$ b $2 + 3i$ c $3 + 2i$

d 6 e $3i$ f $-4 - 3i$ (refer to your sketch)

g $-1 + 2i$ h $2 - 3i$ i $4 - i$

4 For each of these expressions

i simplify by writing them in the form $x + iy$

ii find the modulus and argument.

a $\dfrac{3 + 2i}{1 + 5i}$ b $\dfrac{1}{1 + 3i}$ c $(2 + 4i)(1 - i)$

6 Find the complex number with

a $|z| = 2, \arg z = \dfrac{\pi}{6}$ b $|z| = 3, \arg z = \dfrac{\pi}{4}$ c $|z| = 4, \arg z = \dfrac{\pi}{2}$

d $|z| = 3, \arg z = \dfrac{\pi}{3}$ e $|z| = 2, \arg z = -\dfrac{\pi}{4}$ f $|z| = 1, \arg z = -\dfrac{\pi}{6}$

7 Express each of these complex numbers in polar form. (Give the argument in degrees.)

a $1 + i\sqrt{3}$ b $\sqrt{2} + i\sqrt{2}$ c $-2\sqrt{3} + 2i$

d -1 e $3i$ f $-4 - i4\sqrt{3}$

g $-2\sqrt{3} + 2i$ h $-\sqrt{2} - i\sqrt{2}$ i $-1 - i\sqrt{3}$

Ex. 12.5

1 Simplify these expressions.

a $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

b $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

c $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

Hint: remember $\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

d $\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) \times 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

e $5\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) \times 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

f $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \div 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

g $5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \div 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

h $9\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \div 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

i $8\left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right) \div 2\left(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}\right)$

2 Convert each complex number to polar form then state the product, z_1z_2, and quotient, $\frac{z_1}{z_2}$, of each pair. (Work to 3 significant figures.)

a $z_1 = 3 + 4i, \quad z_2 = 1 + i$

c $z_1 = 1 - i, \quad z_2 = -1 - i$

3 Given that $z = r\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ calculate

a $z^2\left[= r^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2 = r^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$

b $z^3\ [= z^2z]$

c z^4 [remember to bring the argument back into the range $(-\pi, \pi)$]

d z^5 e z^6 f z^7

Answers to AH Maths (MiA), pg. 211, Ex. 12.3

1 **a** $z = i \Rightarrow |z| = 1$ and $\arg z = \frac{\pi}{2}$

b $z = -1 \Rightarrow |z| = 1$ and $\arg z = \pi$

c $z = -i \Rightarrow |z| = 1$ and $\arg z = -\frac{\pi}{2}$

2 Diagrams showing:

a **i** $(3, 4)$ and $(3, -4)$ **ii** $(2, 3)$ and $(2, -3)$

iii $(5, 1)$ and $(5, -1)$

b Reflection in x-axis.

3 **a** Diagram showing $(1, 1)$; $|z| = \sqrt{2}$; $\arg z = \frac{\pi}{4}$

b Diagram showing $(2, 3)$; $|z| = \sqrt{13}$; $\arg z = 0{\cdot}983$ (3 sf)

c Diagram showing $(3, 2)$; $|z| = \sqrt{13}$; $\arg z = 0{\cdot}588$ (3 sf)

d Diagram showing $(6, 0)$; $|z| = 6$; $\arg z = 0$

e Diagram showing $(0, 3)$; $|z| = 3$; $\arg z = \frac{\pi}{2}$

f Diagram showing $(-4, -3)$;
$|z| = 5$; $\arg z = -2.50$ (3 sf)

g Diagram showing $(-1, 2)$; $|z| = \sqrt{5}$; $\arg z = 2{\cdot}03$ (3 sf)

h Diagram showing $(2, -3)$;
$|z| = \sqrt{13}$; $\arg z = -0{\cdot}983$ (3 sf)

i Diagram showing $(4, -1)$;
$|z| = \sqrt{17}$; $\arg z = -0{\cdot}245$ (3 sf)

4 **a** **i** $\frac{1}{2} - \frac{1}{2}i$ **ii** $|z| = \frac{1}{\sqrt{2}}$; $\arg z = -\frac{\pi}{4}$

b **i** $\frac{1}{10} - \frac{3}{10}i$ **ii** $|z| = \frac{1}{\sqrt{10}}$; $\arg z = -1{\cdot}25$

c **i** $6 + 2i$ **ii** $|z| = 2\sqrt{10}$; $\arg z = 0{\cdot}322$

6 **a** $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$

b $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$

c $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4i$

d $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$

e $2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \sqrt{2} - i\sqrt{2}$

f $1\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

7 **a** $2(\cos 60° + i\sin 60°)$ **b** $2(\cos 45° + i\sin 45°)$

c $4(\cos 150° + i\sin 150°)$ **d** $(\cos 180° + i\sin 180°)$

e $3(\cos 90° + i\sin 90°)$ **f** $8(\cos 120° - i\sin 120°)$

g $4(\cos 150° - i\sin 150°)$ **h** $2(\cos 135° - i\sin 135°)$

i $2(\cos 120° - i\sin 120°)$

Jun 1-11:51

Answers to AH Maths (MiA), pg. 215-6, Ex. 12.5

1 **a** $12 \operatorname{cis} \frac{5\pi}{6}$ **b** $10 \operatorname{cis} \frac{5\pi}{12}$ **c** $8 \operatorname{cis} 0$

d $2 \operatorname{cis} \left(-\frac{5\pi}{6}\right)$ **e** $10 \operatorname{cis} \left(-\frac{\pi}{30}\right)$ **f** $2 \operatorname{cis} \left(\frac{\pi}{6}\right)$

g $\frac{5}{2} \operatorname{cis} \left(\frac{\pi}{8}\right)$ **h** $3 \operatorname{cis} \left(\frac{\pi}{3}\right)$ **i** $4 \operatorname{cis} \left(\frac{\pi}{7}\right)$

2 **a** $5 \operatorname{cis} 0{\cdot}927$; $1{\cdot}41 \operatorname{cis} 0{\cdot}785$;
Product: $7{\cdot}07 \operatorname{cis} 1{\cdot}71$; Quotient: $3{\cdot}54 \operatorname{cis} 0{\cdot}142$

c $1{\cdot}41 \operatorname{cis} (-0{\cdot}785)$; $1{\cdot}41 \operatorname{cis} (-2.36)$;
Product: $2 \operatorname{cis} 3.142$; Quotient: $1 \operatorname{cis} (1{\cdot}57)$

3 $z^2 = r^2 \operatorname{cis} \frac{2\pi}{3}$; $z^3 = r^3 \operatorname{cis} \pi$; $z^4 = r^4 \operatorname{cis} \left(-\frac{2\pi}{3}\right)$;
$z^5 = r^5 \operatorname{cis} \left(-\frac{\pi}{3}\right)$; $z^6 = r^6 \operatorname{cis} 0$; $z^7 = r^7 \operatorname{cis} \frac{\pi}{3}$