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Unit 2 : Proof by Mathematical Induction - Lesson 1

Mathematical Induction 1 (Finite Sums)

LI

- Use Proof by Mathematical Induction to solve problems with finite summations.

SC

- Algebra.

Statements, Proofs and Notation

A statement (aka proposition) is a sentence for which it can be decided logically that it is true or false

Examples

- All triangles have 3 sides.
- The current US president is Obama.
- If x is in radians, the derivative of $\sin x$ is $\cos x$.
- If x is in degrees, the derivative of $\sin x$ is $\cos x$.

Non-examples

- Maths is fun.
- Dolphins are smart.
- Aliens exist.
- Some pupils in this AH maths class are VERY annoying.

A proof is a way of showing that a statement is true using the rules of logic

Shorthand

$\forall$ - for all/every

$\exists$ - there exists/is

$\nexists$ - there does not exist

$\in$ - belongs to/is a member of

$\Rightarrow$ - implies

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Proof by Mathematical Induction

The Principle of Mathematical Induction (PMI) states that, to prove a statement (P (n)) about an infinite set of natural numbers :

- Prove the Base Case : $P(n_0)$ is true.
- Prove the Inductive Step :

  $P(k)$ true $\Rightarrow$ $P(k+1)$ true.

Then $P(n)$ is true $\forall\, n \geq n_0$.

Usually, $n_0 = 1$; then we would state the conclusion as 'P (n) is true $\forall\, n \in \mathbb{N}$'.

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Example 1

Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

$$P(n): \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Base Case

Putting $n = 1$ into each side of the $P(n)$ statement gives :

$$\text{LHS} = \sum_{r=1}^{1} r = 1$$

$$\text{RHS} = \frac{1(1+1)}{2} = 1$$

$$\boxed{\text{As LHS} = \text{RHS}, P(1) \text{ is true}}$$

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. assume that :

$$\sum_{r=1}^{k} r = \frac{k(k+1)}{2} \longleftarrow \text{Inductive Hypothesis}$$

We are required to prove (the 'RTP statement') that $P(k+1)$ is true.

RTP statement

$$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$$

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$$

$$\Rightarrow \sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1)$$

$$\Rightarrow \sum_{r=1}^{k+1} r = \frac{k(k+1) + 2(k+1)}{2}$$

$$\Rightarrow \sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$$

$$\boxed{\text{Hence, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true}}$$

'$P(1)$ true' and '$P(k)$ true $\Rightarrow P(k+1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall\, n \in \mathbb{N}$

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Example 2

Prove that $\sum_{r=1}^{n} r.r! = (n+1)! - 1 \quad \forall n \in \mathbb{N}$.

$$P(n): \sum_{r=1}^{n} r.r! = (n+1)! - 1$$

Base Case

$$LHS = \sum_{r=1}^{1} r.r! = 1.1! = 1$$

$$RHS = (1+1)! - 1 = 2! - 1 = 1$$

As LHS = RHS, $P(1)$ is true

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$:

$$\sum_{r=1}^{k} r.r! = (k+1)! - 1 \longleftarrow \text{Inductive Hypothesis}$$

RTP statement

$$\sum_{r=1}^{k+1} r.r! = ((k+1)+1)! - 1$$

$$\sum_{r=1}^{k+1} r.r! = \sum_{r=1}^{k} r.r! + (k+1).(k+1)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r.r! = (k+1)! - 1 + (k+1).(k+1)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r.r! = -1 + (1+k+1).(k+1)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r.r! = -1 + (k+2).(k+1)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r.r! = -1 + (k+2)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r.r! = ((k+1)+1)! - 1$$

Hence, $P(k)$ true $\Rightarrow P(k+1)$ true

'$P(1)$ true' and '$P(k)$ true $\Rightarrow P(k+1)$ true' together imply, by the PMI, that $P(n)$ is true $\forall n \in \mathbb{N}$

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Questions (and 'Answers' !)

Prove by mathematical induction that $\forall\, n \in \mathbb{N}$:

1) $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

2) $$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

3) $$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

4) $$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

5) $$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

6) $$\sum_{r=1}^{n} (8r-5) = 4n^2 - n$$