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Matrices and Systems of Equations - Lesson 1

Gaussian Elimination

LI

- Use Gaussian Elimination to solve a system of 3 equations in 3 variables.

SC

- Elementary Row Operations.
- Primary school arithmetic.

Systems of Equations

A **system of m linear equations in n unknowns** is a collection of linear equations,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

where $a_{ij}, b_i \in \mathbb{R}$.

- Unique solution.
- Infinitely many solutions.
- No solution.

A **consistent system** of linear equations is one that has a unique solution or infinitely many solutions.

An **inconsistent system** of linear equations is one that has no solutions.

Elementary Row Operations and Gaussian Elimination

The **Coefficient Matrix** of the system,

$$\begin{array}{ccccccc} ax & + & by & + & cz & = & j \\ dx & + & ey & + & fz & = & k \\ gx & + & hy & + & iz & = & l \end{array}$$

is the arrangement (aka **matrix**),

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

whereas the **Augmented Matrix** is,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ d & e & f & k \\ g & h & i & l \end{array}\right)$$

Elementary Row Operations on the Augmented Matrix are used to solve the system :

- Multiply a row by a non-zero scalar.
- Add or subtract 2 rows.
- Interchange 2 or more rows.

EROs don't change the solution of the system

We always aim to get zeroes in the following places for the augmented matrix :

$$\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{array}\right)$$

No Solution

A system is inconsistent if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right) \qquad (l \neq 0)$$

Example 1

Show that the system of equations,

$$\begin{array}{ccccccc} x & + & y & + & z & = & 150 \\ x & + & 2y & + & 3z & = & 100 \\ 2x & + & 3y & + & 4z & = & 200 \end{array}$$

has no solution.

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 1 & 2 & 3 & 100 \\ 2 & 3 & 4 & 200 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 1 & 2 & -100 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 0 & 0 & -50 \end{array}\right)$$

Hence, as the LHS is 0 but the RHS in non-zero, there is no solution.

Infinitely Many Solutions

A system has infinitely many solutions if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Example 2

Find the value of k for which the following system has infinitely many solutions, and find the solutions.

$$\begin{aligned} x + 2y + 2z &= 11 \\ x - y + 3z &= 8 \\ 4x - y + kz &= 35 \end{aligned}$$

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & k & 35 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & k-8 & -9 \end{array}\right)$$

$R_3 \to R_3 - 3R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & k-11 & 0 \end{array}\right)$$

For there to be infinitely many solutions, the last row must consist entirely of zeros; hence, $k - 11 = 0 \Rightarrow k = 11$. The remaining 2 equations can be translated into x, y and z form as,

$$\begin{aligned} x + 2y + 2z &= 11 \\ -3y + z &= -3 \end{aligned}$$

Putting $z = t$ into the second equation gives $y = \dfrac{t+3}{3}$ and the first equation then gives, $x = 11 - 2t - \dfrac{2(t+3)}{3} \Rightarrow x = \dfrac{27 - 8t}{3}$.

Hence, the solutions are,

$$\boxed{x = \frac{27 - 8t}{3},\ y = \frac{t+3}{3},\ z = t \quad (\forall t \in \mathbb{R})}$$

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Unique Solution

A system has a unique solution if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right) \qquad (i \neq 0)$$

Example 3

Find the solution to the system,

$$\begin{aligned} x + y + z &= 2 \\ 4x + 2y + z &= 4 \\ x - y + z &= 4 \end{aligned}$$

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right)$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -4 \\ 0 & -2 & 0 & 2 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & -4 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

In terms of the unknowns, this is,

$$\begin{aligned} x + y + z &= 2 \\ -2y &= 2 \\ -3z &= -6 \end{aligned}$$

The third and second equations respectively give $z = 2$ and $y = -1$. Back-substitution into the first equation then gives $x = 2 - 2 + 1 = 1$. Hence, the unique solution is $\boxed{x = 1, y = -1, z = 2}$

AH Maths - MiA (2^{nd} Edn.)

- pg. 265-6 Ex. 14.4 Q 1 - 3.
- pg. 268 Ex. 14.6 Q 1 - 4.

Ex. 14.4

1 For each system of equations

i express it as an augmented matrix

ii reduce the matrix, using row operations, to upper triangular form

iii using back-substitution, work out the values of the variables.

a
$$\begin{aligned} x + 2y + z &= 8 \\ 3x + y - 2z &= -1 \\ x + 5y - z &= 8 \end{aligned}$$

b
$$\begin{aligned} 2x + 3y - z &= -1 \\ x - 3y - 2z &= 4 \\ 5x + y + 3z &= 4 \end{aligned}$$

c
$$\begin{aligned} 3x + y \phantom{{}+3z} &= 5 \\ x + 2y - 3z &= -12 \\ x + 2z &= 10 \end{aligned}$$

d
$$\begin{aligned} 3x - 4y + z &= 24 \\ x - 2y - 2z &= 7 \\ x + y + z &= 4 \end{aligned}$$

e
$$\begin{aligned} 4x + 2y + z &= 3 \\ x + 3y + 5z &= 3 \\ 2x + 3z &= 5 \end{aligned}$$

f
$$\begin{aligned} x + y + 5z &= 0 \\ 4x + y - 6z &= -17 \\ x - y - z &= 0 \end{aligned}$$

2 A parabola passes through the points (1, 2), (2, 7) and (3, 14).
It has an equation of the form $y = ax^2 + bx + c$.

a Use the information to form a 3×3 system of equations.

b Solve the system by Gaussian elimination.

c Write the equation of the parabola.

3 An archaeological dig discovers the remains of a circular Roman amphitheatre. Using a suitable set of axes and convenient units, the archaeologists positively identify three points on its circumference: $(-2, -1)$, $(-1, 2)$ and $(6, 3)$.

a Assuming the perimeter has an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ form a system of equations in g, f and c.

b Solve the system and identify the equation of the perimeter.

c What is the radius of the amphitheatre?

Ex. 14.6

1 Attempt to reduce each of these systems of equations to upper triangular form.

- Where this is possible quote the unique solution.
- Where there is a redundant equation, find a general solution.
- Where there is inconsistency, declare that there are no solutions.

a $\begin{aligned} 3x + 2y + 5z &= 0 \\ 2x + y - 2z &= 5 \\ 7x + 4y + z &= 10 \end{aligned}$

b $\begin{aligned} x + y - z &= 4 \\ 2x - y + 2z &= -2 \\ x - 3y - 4z &= -1 \end{aligned}$

c $\begin{aligned} 2x - y + 3z &= 6 \\ x + y + 2z &= 7 \\ 4x + y + 7z &= 9 \end{aligned}$

d $\begin{aligned} 2x - 3y + z &= 2 \\ x + y - 3z &= 7 \\ 5x - 2y - z &= 14 \end{aligned}$

e $\begin{aligned} x + 2y - z &= 3 \\ x + 3z &= 5 \\ 4x + 2y + 8z &= 10 \end{aligned}$

f $\begin{aligned} 5x - 3y - z &= -12 \\ 2x + y + 3z &= 3 \\ 20x - y + 13z &= -9 \end{aligned}$

2 The system of equations

$$\begin{aligned} x + 2y - z &= 8 \\ 3x + y + 2z &= -1 \\ x + y + kz &= -6 \end{aligned}$$

has no solutions. What is the value of k?

3 Find the value of k that makes the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y - 2z &= -1 \\ x - y + kz &= 7 \end{aligned}$$

have infinitely many solutions.

4 For what values of d and e will the three equations

$$\begin{aligned} x + 3y - 2z &= 8 \\ 2x + y - 3z &= 5 \\ 7x - 4y + dz &= e \end{aligned}$$

have a no solution

b infinitely many solutions

c a unique solution?

Answers to AH Maths (MiA), pg. 265-6, Ex. 14.4

1 a $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 3 & 1 & -2 & -1 \\ 1 & 5 & -1 & 8 \end{array}\right); \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right); \begin{array}{l} x = 1 \\ y = 2 \\ z = 3 \end{array}$

b $\left(\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 1 & -3 & -2 & 4 \\ 5 & 1 & 3 & 4 \end{array}\right); \left(\begin{array}{ccc|c} 1 & 1\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & 1 & 0 \end{array}\right); \begin{array}{l} x = 1 \\ y = -1 \\ z = 0 \end{array}$

c $\left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 1 & 2 & -3 & -12 \\ 1 & 0 & 2 & 10 \end{array}\right); \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & 0 & 1\frac{2}{3} \\ 0 & 1 & -1\frac{4}{5} & -8\frac{1}{5} \\ 0 & 0 & 1 & 4 \end{array}\right); \begin{array}{l} x = 2 \\ y = -1 \\ z = 4 \end{array}$

d $\left(\begin{array}{ccc|c} 3 & -4 & 1 & 24 \\ 1 & -2 & -2 & 7 \\ 1 & 1 & 1 & 4 \end{array}\right); \left(\begin{array}{ccc|c} 1 & -1\frac{1}{3} & \frac{1}{3} & 8 \\ 0 & 1 & 3\frac{1}{2} & 1\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right); \begin{array}{l} x = 5 \\ y = -2 \\ z = 1 \end{array}$

e $\left(\begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ 1 & 3 & 5 & 3 \\ 2 & 0 & 3 & 5 \end{array}\right); \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 1\frac{9}{10} & \frac{9}{10} \\ 0 & 0 & 1 & 1 \end{array}\right); \begin{array}{l} x = 1 \\ y = -1 \\ z = 1 \end{array}$

f $\left(\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 4 & 1 & -6 & -17 \\ 1 & -1 & -1 & 0 \end{array}\right); \left(\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 0 & 1 & 8\frac{2}{3} & 5\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right); \begin{array}{l} x = -2 \\ y = -3 \\ z = 1 \end{array}$

2 a $a + b + c = 2, 4a + 2b + c = 7, 9a + 3b + c = 14$

b $a = 1, b = 2, c = -1$

c $y = x^2 + 2x - 1$

3 a $-4g - 2f + c = -5, -2g + 4f + c = -5,$
$12g + 6f + c = -45$

b $g = -3, f = 1, c = -15; x^2 + y^2 - 6x + 2y - 15 = 0$

c $r = 5$

Answers to AH Maths (MiA), pg. 268, Ex. 14.6

1 **a** $x = 9t + 10, y = -16t - 15, z = t$ **b** $x = 1, y = 2, z = -1$

c Inconsistent: no solutions.

d $x = 3, y = 1, z = -1$

e Inconsistent: no solutions **f** $x = -(8t + 3)/11, y = -(17t - 39)/11, z = t$

2 $k = 0$

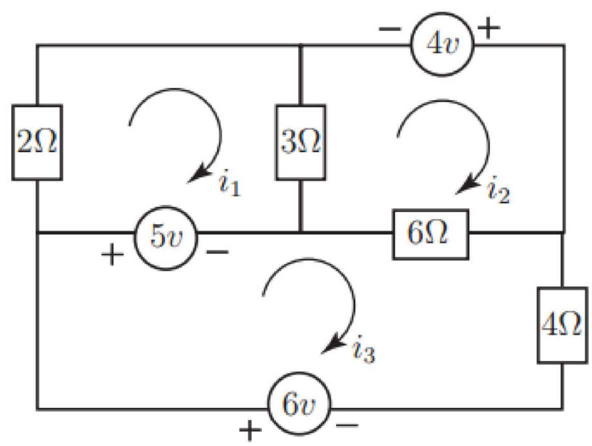
3 $k = 9$

4 **a** $d = -9$ and $e \neq 1$ **b** $d = -9$ and $e = 1$

c $d \neq -9$

Real-Life Application (Kirchoff's Laws)

In the circuit shown find the currents (i_1, i_2, i_3) in the loops.



<u>KCL :</u> The sum of all currents going into a node equals the sum of all currents leaving the node.

<u>KVL :</u> In any loop around a circuit, the sum of the emfs equals the sum of the potential differences.

For the above circuit, show that :

$$i_1 = \frac{34}{15} \qquad i_2 = \frac{19}{9} \qquad i_3 = \frac{41}{30}$$